The longest shortest piercing.

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Abstract: *We generalize an old problem and its partial solution of Pólya [7] to the n-dimensional setting. Given a plane domain $\Omega \subset \mathbb{R}^2$, Pólya asked in 1958 for the shortest bisector of Ω, that is for the shortest line segment $l(\Omega)$ which divides Ω into two subsets of equal area. He claimed that among all centrosymmetric domains of given area $l(\Omega)$ becomes longest for a disk. His proof, however, does not seem to be valid for domains that are not starshaped with respect to the center of Ω. In the present note we provide two proofs that it suffices to restrict attention to starshaped sets. Moreover we state and prove a related inequality in $\mathbb{R}^n$. Given the volume of a measurable set Ω with finite Lebesgue measure, only a ball centered at zero maximizes the length of the shortest line segments running through the origin. In this sense the ball has the longest shortest piercing.*

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1 Motivation and Result

Let Ω be a measurable (possibly unbounded) set with finite volume in $\mathbb{R}^n$, $n \geq 2$. For every z on the unit sphere $\mathbb{S}^n$ and ray L_z emanating from the origin and passing through z we can measure $\ell_z(\Omega)$, the length of $L_z \cap \Omega$, which can be given by

$$\ell_z(\Omega) = \int_0^\infty \chi_\Omega(z, r)dr, \tag{1}$$

where $\chi_\Omega(z,r)$ is the characteristic function of the set $L_z \cap \Omega$. Note that $\ell_z(\Omega)$ can be infinite for some $z \in \mathbb{S}^n$, but since the volume of Ω is finite, this can happen only on a nullset of $\mathbb{S}^n$. We define the piercing length of Ω as

$$\ell(\Omega) := \inf_{z \in \mathbb{S}^n} (\ell_z(\Omega) + \ell_{-z}(\Omega)) \tag{2}$$

and prove the following result:

Theorem 1 *Let Ω be a measurable (possibly unbounded) set with finite volume in $\mathbb{R}^n$, $n \geq 2$. Then the inequality*

$$\ell(\Omega) \leq \ell(\Omega^*), \tag{3}$$

holds, where Ω^* denotes the ball centered at the origin and of the same volume as Ω. Moreover, equality holds in (3) if and only if $\Omega = \Omega^*$ modulo a set of measure zero.

For the case $n = 2$ this theorem was implicitly stated in [7], but the proof given there had starshaped centrosymmetric domains in mind. We call a set Ω **centrosymmetric** (with respect to the origin) iff $x \in \Omega$ implies $-x \in \Omega$ and **starshaped** (with respect to the origin) iff $x \in \Omega$ and $t \in [0,1]$ imply $tx \in \Omega$. Later Cianchi gave an independent proof of (3) for convex centrosymmetric plane domains, see Theorem 4 in [2].

2 Proof

First we prove the theorem for starshaped centrosymmetric sets. In a second step we show that the maximum of $\ell(\Omega)$ over all starshaped sets is assumed among centrosymmetric sets. In a third step we show that the maximum of $\ell(\Omega)$ over all measurable sets with finite n-dimensional Lebesgue measure is necessarily attained among starshaped sets.

Step 1. Suppose Ω is an arbitrary centrosymmetric starshaped (possibly unbounded) set with finite volume in $\mathbb{R}^n$ but not a ball (modulo a nullset). Then there must exist a boundary point x of Ω which lies in the interior of Ω^*, and so does $-x$. Therefore, the line segment connecting x with $-x$ is strictly shorter than the diameter of Ω^*, that is $\ell(\Omega) < \ell(\Omega^*)$. For $n = 2$ this is Pólya's proof, but it extends without changes to general $n \geq 2$.

Step 2. We will prove that *if* Ω *is starshaped but not centrosymmetric, then it can be replaced by a centrosymmetric starshaped set* $\tilde{\Omega}$ *of same volume as* Ω *such that* $\ell(\Omega) \leq \ell(\tilde{\Omega})$. In fact, if we replace the representation of Ω in polar coordinates $\ell_z(\Omega)$ by $\tilde{\ell}_z(\Omega)$ with

$$\left(\frac{1}{n}(\ell_z^n + \ell_{-z}^n)\right)^{1/n} = \left(\frac{2}{n}\tilde{\ell}_z^{\,n}\right)^{1/n}, \tag{4}$$

then the set $\tilde{\Omega}$ whose boundary is described by $\ell_z(\tilde{\Omega}) := \tilde{\ell}_z(\Omega)$ is of same volume as Ω. Its piercing length, however, has not decreased, because the convexity of the mapping $t \mapsto t^n$ implies

$$\left(\tilde{\ell}_z(\Omega)\right)^n = \frac{1}{2}(\ell_z^n + \ell_{-z}^n) \geq \left(\frac{\ell_z(\Omega) + \ell_{-z}(\Omega)}{2}\right)^n \geq (\ell(\Omega))^n ,$$

and after infimizing over $z \in \mathbb{S}^n$ we arrive at

$$\ell(\tilde{\Omega}) \geq \ell(\Omega)$$

as claimed.

Step 3. We will prove the following *claim*: *If* Ω *is not starshaped, then it can be replaced by a starshaped set* $\Omega^\#$ *of same volume as* Ω *such that* $\ell(\Omega) \leq \ell(\Omega^\#)$. This claim and Steps 1 and 2 result in a proof of Theorem 1.

For an arbitrary (possibly unbounded) set Ω with finite volume in $\mathbb{R}^n$ its volume is given by

$$|\Omega| = \int_{\mathbb{S}^n} \int_0^\infty \chi_\Omega(z,r) r^{n-1} dr \; dz$$

in polar coordinates. Let us consider the function

$$R_\Omega^\#(z) = \left[n \int_0^\infty \chi_\Omega(z,r) r^{n-1} dr \right]^{\frac{1}{n}} \tag{5}$$

which can be infinite for some $z \in \mathbb{S}^n$, but since the volume of Ω is finite, only on a nullset of $\mathbb{S}^n$. If $\Omega^\#$ is defined (modulo a nullset) as the starshaped set bounded in polar coordinates by $R_\Omega^\#(z)$, then $|\Omega| = |\Omega^\#|$, that is Ω has been rearranged by starshaped rearrangement into an equimeasurable starshaped set, see [6]. What happens to $\ell(\Omega)$ after this operation? First, it is clear that the inequality

$$\ell(\Omega) \leq \ell_z(\Omega) + \ell_{-z}(\Omega) \tag{6}$$

holds for all $z \in \mathbb{S}^n$. Second, due to monotonicity of the function $\rho(r) = r^{n-1}$, $n \geq 2$, we have the inequality

$$\int_0^{\ell_z(\Omega)} r^{n-1} dr \leq \int_0^\infty \chi_\Omega(z,r) r^{n-1} dr$$

and thus the inequality

$$\frac{1}{n} \left(\ell_z(\Omega)\right)^n \leq \frac{1}{n} \left(R_\Omega^\#(z)\right)^n, \tag{7}$$

which holds for all $z \in \mathbb{S}^n$. In turn, (6) and (7) yield the inequalities

$$\ell(\Omega) \leq \ell_z(\Omega) + \ell_{-z}(\Omega) \leq R_\Omega^\#(z) + R_\Omega^\#(-z) = \ell_z(\Omega^\#) + \ell_{-z}(\Omega^\#) \tag{8}$$

for all $z \in \mathbb{S}^n$. Finally, after infimizing (8) over all $z \in \mathbb{S}^n$ we obtain

$$\ell(\Omega) \leq \ell(\Omega^\#)$$

which shows that the piercing length of Ω does not decrease in passing from Ω to $\Omega^\#$. Since we try to maximize the domain functional $\ell(\Omega)$, it suffices to study starshaped sets.

For a *second* proof of Step 3 we can also follow the idea in [3], there for the case $n = 2$, and recall a Hardy-Littlewood inequality that seems to be mathematical folklore. If u and v are two nonnegative functions defined on $\mathbb{R}_+$, and if u^* denote the decreasing and v_* the increasing rearrangement of u and v, then

$$\int_0^\infty u(r)v(r)\ dr \geq \int_0^\infty u^*(r)v_*(r)\ dr.$$

For the benefit of the reader let us remark in passing that its proof goes along the lines of Lemma 2.1 in [6] by reduction to the product of two nonnegative finite sequences. This product becomes minimal when the sequences are oppositely ordered, see Theorem 368 in [5].

Identifying u with $\chi_\Omega(z,r)$ and v with r^{n-1} gives now

$$|\Omega| = \int_{\mathbb{S}^n} \int_0^\infty \chi_\Omega(z,r) r^{n-1}\ drdz \geq \int_{\mathbb{S}^n} \int_0^\infty \chi_\Omega^*(z,r) r^{n-1}\ drdz = |\tilde{\Omega}|,$$

i.e. the volume of a starshaped set $\tilde{\Omega}$ whose characteristic function is given by $\chi^*_{\tilde{\Omega}}(z,r)$. While the percing length $\ell(\Omega)$ remains invariant under this rearrangement, in fact $\ell(\Omega) = \ell(\tilde{\Omega})$ by construction, the volume decreases, unless Ω was already starshaped. Now we define $\Omega^{\#}$ to be a rescaled (enlarged) version of $\tilde{\Omega}$, so that $|\Omega^{\#}| = |\Omega|$. Then again $\ell(\Omega) \leq \ell(\Omega^{\#})$ as claimed.

3 Related questions

In this section we address related questions.

A) We have learned from F. Brock that he and M.Willem have considered planes which cut centrosymmetric n-dimensional bodies into two halves of equal volume. If $A_{n-1}(\Omega)$ denotes a cut through Ω which minimizes $(n-1)$-dimensional area, they were able to show that $A_{n-1}(\Omega) \leq A_{n-1}(\Omega^*)$.

B) If one wants to trade the assumption of centrosymmetry against convexity, already in 2 dimensions the question of the longest shortest cut that bisects area poses a major challenge. If one allows only straight lines to cut a convex plane set Ω into two parts of equal area (and any straight line can be shifted to do so), then among all (convex plane) sets of given area, the length A_1 of the bisecting line segment becomes maximal *not for the disk* Ω^*, but for the so-called Auerbach triangle T, see [3]. The Auerbach triangle belongs to a class of so-called Zindler sets. By definition a Zindler set Z has the remarkable property that every line-segment which bisects the area of Z has the same length. To be precise [3] contains a proof that

$$A_1(\Omega) \leq A_1(T) \tag{9}$$

for every plane convex set Ω of given area, while [4] proves (9) for the smaller class of plane convex Zindler sets.

On the other hand, the shortest curve that bisects the area of the Auerbach triangle, is a circular arc and its length is shorter than the diameter of the disc of equal area. In fact, it has long been conjectured that the among all plane convex sets of given area, the disk maximizes length of the shortest curve that bisects the area. In [3] this conjecture is confirmed by a long and rather technical proof.

C) The result of Brock and Willem described above under A) as well as our Theorem 1 can be generalized to the k-dimensional setting for all $k = 1, \ldots, n-1$. Then it reads as follows: If $A_k(\Omega)$ denotes a k-dimensional cut (or generalized piercing) through Ω which minimizes k-dimensional area, then $A_k(\Omega) \le A_k(\Omega^*)$. To prove these results by induction with respect to k one can follow Steps 1–3: First, using an analytic version of Pólya's proof and an iteration from $n-k+1$ to $n-k$ for any $k = 1, \dots, n-1$, one can obtain the corresponding $(n-k)$-dimensional results for starshaped centrosymmetric sets. Next, using Step 2, one can show that the maximum of $(\ell(\Omega))^k$ is assumed among centrosymmetric sets. Finally, one can show that the maximum of $A_k(\Omega)$ over all measurable sets with finite n-dimensional Lebesgue measure is necessarily attained among starshaped sets by using the following observation which can be of independent interest. To formulate the corresponding result, for any $z \in \mathbb{S}^n$ and any $1 \le p \le n$ let us consider the function

$$R_\Omega(z,p) = \left[p \int_0^\infty \chi_\Omega(z,r) r^{p-1} dr \right]^{\frac{1}{p}}. \tag{10}$$

Lemma 1 *Let Ω be a measurable (possibly unbounded) domain with finite volume in $\mathbb{R}^n$, $n \ge 2$. Then the inequality*

$$R_\Omega(z,p) \le R_\Omega(z,q) \tag{11}$$

holds for any $z \in \mathbb{S}^n$ and any $1 \le p < q \le n$. Moreover, equality in (11) holds if and only if Ω is a starshapped set.

Proof. Let $r = \rho^{1/q}$. Then by (10),

$$[R_\Omega(z,p)]^p = p \int_0^\infty \chi_\Omega(z,r) r^{p-1} dr = \frac{p}{q} \int_0^\infty \chi_\Omega(z, \rho^{1/q}) \rho^{\frac{p}{q}-1} d\rho. \tag{12}$$

It is clear that in the integral on the right-hand side of (12) one integrates over the set $\mathcal{L}_z \subset L_z$ of length

$$|\mathcal{L}_z| = \int_0^\infty \chi_\Omega(z, \rho^{1/q}) d\rho.$$

Changing in (12) the variable of integration to $\rho = r^q$, we have the relations

$$|\mathcal{L}_z| = \int_0^\infty \chi_\Omega(z, \rho^{1/q}) d\rho = q \int_0^\infty \chi_\Omega(z,r) r^{q-1} dr = [R_\Omega(z,q)]^q. \tag{13}$$

Further, due to the fact that the function $f(\rho) = \rho^{\frac{p}{q}-1}$ decreases monotonically on $\mathbb{R}_+$, we conclude from (12) by (13) that

$$\frac{p}{q} \int_0^\infty \chi_\Omega(z, \rho^{1/q}) \rho^{\frac{p}{q}-1} d\rho \le \frac{p}{q} \int_0^{(R_\Omega(z,q))^q} \rho^{\frac{p}{q}-1} d\rho. \tag{14}$$

Integrating on the right-hand side of (14) we obtain the inequality

$$\frac{p}{q}\int_0^\infty \chi_\Omega(z,\rho^{1/q})\rho^{\frac{p}{q}-1}d\rho \le (R_\Omega(z,q))^p$$

which, together with (12), yields (11). Equality in (11) holds iff one has equality in (14), which in turn implies

$$R_\Omega(z,q) = \left[q\int_0^{\ell_z(\Omega)} r^{q-1}dr\right]^{\frac{1}{q}},$$

i.e., iff the set Ω under consideration is already starshaped. Finally let us remark that a discrete version of this Lemma can be found in [6], p.64.

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