A study on gradient blow up for viscosity solutions of fully nonlinear, uniformly elliptic equations

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Abstract

We investigate sharp conditions for boundary and interior gradient estimates of continuous viscosity solutions to fully nonlinear, uniformly elliptic equations under Dirichlet boundary conditions. When these conditions are violated, there can be blow up of the gradient in the interior or on the boundary of the domain. In particular we derive sharp results on local and global Lipschitz continuity of continuous viscosity solutions under more general growth conditions than before. Lipschitz regularity near the boundary allows us to predict when the Dirichlet condition is satisfied in a classical and not just in a viscosity sense, where detachment can occur. Another consequence is this: if interior gradient blow up occurs, Perron-type solutions can in general become discontinuous, so that the Dirichlet problem can become unsolvable in the class of continuous viscosity solutions.

Keywords: fully nonlinear elliptic equations, viscosity solutions, gradient estimates, gradient blow up

*This paper is dedicated to Constanine Dafermos, teacher and friend

1 Introduction

In the theory of viscosity solutions for fully nonlinear elliptic Dirichlet problems

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega, \tag{1.1}$$
$$u = \psi(x) \quad \text{on } \partial\Omega, \tag{1.2}$$

introduced by Crandall and Lions [17], the main issues are existence, uniqueness and stability of viscosity solutions in $C(\overline{\Omega})$. Here $\Omega \subset \mathbb{R}^n$ is a bounded domain and $F \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^n)$, and S^n is the space of all symmetric $n \times n$ matrices. The theory of viscosity solution was systematically developped under very weak structural assumptions on F, such as degenerate ellipticity

$$F(x, r, p, X) \leq F(x, r, p.Y) \tag{1.3}$$

for any $x \in \overline{\Omega}$, $r \in \mathbb{R}$, $p \in \mathbb{R}^n$, $X, Y \in S^n$ with $X \geq Y$,
and strict monotonicity: there exists a constant $c_0 > 0$ such that

$$c_0(r - s) \leq F(x, r, p, X) - F(x, s, p.X)) \tag{1.4}$$

for any $x \in \overline{\Omega}$, $r \geq s \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $X \in S^n$, in [18, 35, 36, 37, 39, 40, 41, 62] and [70], see the references in [18]. As for higher regularity of viscosity solutions of the equation $F(D^2u, x) = f(x)$, interior $C^{1,\alpha}$, C^2 and local $W^{2,p}$ estimates were proved by Caffarelli [14]. In subsequent developments [73, 56, 61] interior C^α and $C^{1,\alpha}$ estimates were derived for viscosity solutions of more general fully nonlinear uniformly elliptic and parabolic equations, however under the assumption that $F(x, r, p, 0)$ has at most quadratic growth in $|p|$. In [71] Trudinger proved interior C^α and $C^{1,\alpha}$ estimates for viscosity solutions of uniformly elliptic equations (1.1) (1.2) under quadratic growth with respect to $|p|$ for the lower order term $F(x, r, p, 0)$ and under additional structure conditions for the growth of $|F(x, r, p, X) - F(x, t, q, Y)|$ in p, q, X and Y. In all these papers Lipschitz estimates were obtained by interpolation. We should also mention L^p-viscosity solutions and their equivalence with other weak notions of solutions, see [16, 19, 20]. For L^p viscosity solutions interior $W^{1,p}$ estimates were derived in [65, 47] and global $W^{1,p}$ estimates in [75] for uniformly elliptic equations with at most quadratic growth of $F(x, r, p, 0)$ in $|p|$. In contrast to all these papers we allow for superquadratic growth in $|p|$.

The aim of this paper is to identify sharp structural conditions that seperate the possibility for gradient estimates from that of gradient blow up for viscosity solutions of (1.1) (1.2). Our motivation to study this question is the crucial role that gradient estimates play in existence proofs. Gradient estimates guarantee that the Dirichlet condition is satisfied in the classical sense, while their violation means that (1.2) is merely satisfied in the viscosity sense, see Def. 2.2. If interior gradient blow up occurs, we conjecture that Perron's solutions should (in general) be discontinuous and that the comparison princple for semicontinuous viscosity subsolutions fails (see Example 2.14).

In fact, even for classical solutions the theory of gradient estimates is far from its final state, in particular for quasilinear and fully nonlinear equations. For linear, uniformly elliptic equations the classical solvability of the Dirichlet problem is well known to depend only on the smoothness of its coefficients, on the underlying domain Ω and the data of the problem. For nonlinear equations, however, boundary gradient estimates are an essential step in proving the existence a) of classical solutions via Schauder's fixed point theorem, and b) of continuous viscosity solutions in $C(\overline{\Omega})$ and their Lipschitz and $C^{1,\alpha}$ regularity when they satisfy the Dirichlet condition in the classical sense, see [15, 31, 48]. In the beginning of the 20th century it became clear from Hilbert's 19th and 20th problem that such estimates are indispensable. At this time Bernstein [9, 10] derived gradient estimates for nonlinear equations in plane domains in his fundamental papers [9, 10].

Half a century later Serrin extended Bernstein's results in his elegant paper [59] to quasilinear equations in higher dimensions and clarified the important role of the geometry of $\partial\Omega$ in this context. The present state of the art on gradient estimates of classical solutions to quasilinear equations seems to be contained in results of Ladyshenskaya and Ural'tseva [49, 50], Gilbarg and Trudinger [31], Krylov [48], Evans [22], Caffarelli and Cabré [15] and others, see the references therein. Let us also mention the paper of Barles [6] where the classical Bernstein method was applied and extended to viscosity solutions of (1.1). Moreover, Barles simplified some of the sufficient conditions for gradient estimates.

Bernstein also observed that his barrier functions can sometimes be used to derive nonexistence results for classical solutions of Dirichlet problems. It is only now, that we have the theory of viscosity solutions at our disposal, that we can explain these phenomena as a "detachment" from the given Dirichlet data that can be reconciled with the notion of "Dirichlet condition in the viscosity sense". In this respect our results revisit Bernstein's method for viscosity solutions.

In particular we derive sharp conditions for Lipschitz estimates of continuous viscosity solutions to (1.1) (1.2). To be specific, for estimates near the boundary, these conditions involve the exact growth rate of $F(x, r, p, 0)$ at infinite p. If this growth is of order $|p|^2 \ln(1+|p|)$, the superquadratic growth must be compensated by suitable boundary behaviour as explained in Section 4. Otherwise there can be gradient blow up. Our results (in particular Theorem 2.3) not only recover but also improve Theorem 4.1 and Remark 4.3 of [38], because we do not require strict convexity of Ω

There have also been detailed studies on nonexistence of classical solutions for Dirichlet problems involving the mean curvature operator [28, 32, 60, 74]. We wish to point out, that this operator is of divergence type and not uniformly elliptic. In contrast to the mean curvature operator our operators have in general no variational structure and are assumed uniformly elliptic.

In the present paper we consider uniformly elliptic equations, i.e. we assume that there exist positive constants c_1 and c_2 such that

$$c_1\ tr(X-Y) \leq F(x,r,p,Y) - F(x,r,p.X) \leq c_2\ tr(X-Y). \tag{1.5}$$

for any $x \in \overline{\Omega},\ r \in \mathbb{R},\ p \in \mathbb{R}^n, X, Y \in S^n$ with $X \geq Y$. Moreover we assume that $F(x,r,p,X)$ is locally Lipschitz-continuous in the gradient variable p, i.e. for any $K \geq 0$ there exists a constant $C_K \geq 0$ such that

$$|F(x,r,p,X) - F(x,r,q,X)| \leq C_K|p-q| \tag{1.6}$$

for any $x \in \overline{\Omega},\ r \in \mathbb{R},\ p \in \mathbb{R}^n, X \in S^n$ with $|r|,\ |p|,\ |q|$ and $||X|| \leq K$. Because of the strong ellipticity we can weaken assumption 1.4 and allow $c_0 = 0$ so that $F(x,r,p,X)$ is monotone nondecreasing in r, i.e. (1.4) holds with $c_0 = 0$:

$$F(x,r,p,X) \leq F(x,s,p.X) \quad \text{for any } x \in \overline{\Omega},\ r \leq s \in \mathbb{R},\ p \in \mathbb{R}^n, X \in S^n. \tag{1.7}$$

As shown in [43], conditions (1.5), (1.6) and (1.7) guarantee the validity of a strong interior and boundary maximum principle for semicontinuous viscosity subsolutions, and of a comparison principle between viscosity sub- and supersolutions, provided one of them is a classical one. One can even weaken (1.5) to condition (8) in [43], i.e. c_1 and c_2 can depend on K. So once we can construct suitable and explicit C^2 functions that are viscosity sub- and supersolutions, the comparison principle Theorems 1 to 3 in [43] can provide barriers and leads to gradient estimates for continuous viscosity solutions of (1.1) in the interior and on the boundary.

Our paper is organized as follows. In Section 2 we give definitions and present our main results. In Section 3 we put our paper the context of related results and comment on those. Section 4 contains proofs of the boundary estimates and deals with gradient blow-up on the boundary, while Section 5 treats interior estimates and interior gradient blow-up.

Our results for gradient a priori estimates and gradient blow up seem to cover all previous results, which are basically for equations with natural growth conditions of the lower order term, i.e., quadratic growth with respect to the gradient variables or $\beta = 0$ in (2.7)–(2.9). Moreover, for equations with superquadratic growth in the lower order term and $\beta > 0$ in (2.8),(2.9) we show the important role of the geometry of the domain and the boundary data. Such type of boundary gradient estimates are well-known only for geometric equations, Monge-Ampere type equations (Minkowski problem) and mean curvature type equations (see [23, 29, 31, 28, 32, 59, 72]) but not for general elliptic equations. Our results are sharp in the refined power-log scale of the nonlinearities of the equations. The threshhold between the gradient estimates and gradient blow up of the viscosity solutions has now been clarified via clear and explicit criteria. In contrast to previous papers on blow up, which are incidental and for some special equations, we suggest a general theory that explains gradient blow up phenomena from a different viewpoint.

2 Definitions and Main Results

Throughout the paper we use the notation and definitions from [18].

Definition 2.1. *Suppose that* $F \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^n)$ *satisfies (1.3) (1.7). An upper semicontinuous function* $u \in USC(\Omega)$ *is a* viscosity subsolution *of (1.1) if* $F(x_0, u(x_0), p, X) \leq 0$ *for every* $x_0 \in \Omega$ *and for every* $(p, X) \in \mathcal{J}^{2,+}_{\Omega}(u(x_0))$. *Here the second order* superjet $\mathcal{J}^{2,+}_{\Omega}(u(x_0))$ *is defined as the set of those* $(p, X) \in \mathbb{R}^n \times S^n$ *for which*

$$u(x) \leq u(x_0) + \langle p, x - x_0 \rangle + \frac{1}{2}\langle X(x - x_0), x - x_0 \rangle + o(|x - x_o|^2) \tag{2.1}$$

as $x \to x_0$.

Analogously we define lower semicontinuous viscosity supersolutions $v \in LSC(\Omega)$ *of (1.1) by means of the reverse inequality* $F(y_0, v(y_0), q, Y) \geq 0$ *for every* $y_0 \in \Omega$ *and for every* $(q, Y) \in \mathcal{J}^{2,-}_{\Omega}(v(y_0))$, *where* $\mathcal{J}^{2,-}_{\Omega}(v(y_0)) = \mathcal{J}^{2,+}_{\Omega}(-v(y_0))$.

Finally, we call a continuous function $u : \Omega \to \mathbb{R}$ *a* viscosity solution *of (1.1) if it is both a viscosity sub- and supersolution.*

Throughout the paper we consider only Dirichlet boundary condtions (1.2) on C^2-smooth domains and with data ψ that are traces of C^2-functions. We should explain why we make these smoothness assumptions. It is well known that even classical solutions of linear elliptic equations are in general not Lipschitz continuous on $\partial\Omega$ when the Dirichlet data are only of class C^1, see [52], or when $\partial\Omega$ is not smooth. In this paper we want to investigate only gradient blow up phenomena caused by the nonlinearity of the equation and the geometry of the domain, but not by the lack of smoothness of ψ or $\partial\Omega$. We say that the Dirichlet condition is satisfied in the classical sense, if $u \in C(\overline{\Omega})$ and if $u(x) = \psi(x)$ for every $x \in \partial\Omega$. Unfortunately the classical Dirichlet problem is not stable under small perturbations of the differential equation, see Section 7 in [18] or [42]. Therefore one has to weaken condition (1.2) so that it can accomodate small perturbations. We recall Definition 7.4 in [18].

Definition 2.2. *Suppose* $F \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^n)$ *satisfies (1.3) (1.7). An upper semicontinuous function* $u \in USC(\overline{\Omega})$ *is called a* viscosity subsolution of the Dirichlet problem *(1.1) (1.2) if u is a viscosity subsolution of the differential equation (1.1) in* Ω *and if* $\min\{F(x_0, u(x_0), p, X), u(x_0) - \psi(x_0)\} \leq 0$ *for every* $x_0 \in \partial\Omega$ *and* $(p, X) \in \mathcal{J}^{2,+}_{\overline{\Omega}}(u(x_0))$.

Similarly $u \in LSC(\overline{\Omega})$ *is called a* viscosity supersolution of the Dirichlet problem *(1.1) (1.2) if u is a viscosity supersolution of the differential equation (1.1) in* Ω *and if* $\max\{F(y_0, u(y_0), p, X), u(y_0) - \psi(v_0)\} \geq 0$ *for every* $x_0 \in \partial\Omega$ *and* $(p, X) \in \mathcal{J}^{2,-}_{\overline{\Omega}}(u(y_0))$.

Finally, a function $u \in C(\overline{\Omega})$ *is called a* viscosity solution of the Dirichlet problem *(1.1) (1.2), if u is a viscosity sub- and supersolution of the Dirichlet problem (1.1) (1.2).*

In order to formulate our results we denote by φ and ϕ continuous functions from $[0, \infty)$ to $[0, \infty)$, which are nondecrasing, positive on $(0, \infty)$ and satisfy

$$\int_1^\infty \frac{dt}{t\varphi(t)} = \infty \tag{2.2}$$

and

$$\int_1^\infty \frac{dt}{t\phi(t)} < \infty. \tag{2.3}$$

Examples are given by $\varphi(t) = \ln(1+t)$ and $\phi(t) = \ln^{1+\varepsilon}(1+t)$ with $\varepsilon > 0$.

Let us denote by $d(x)$ the distance function to the boundary $d(x) := \operatorname{dist}(x, \partial\Omega)$. Since $\partial\Omega$ is of class C^2, d is of class $C^2(\Omega_0)$ in a sufficiently small neighborhood $\Omega_0 := \{x \in \Omega;\ d(x) < d_0\}$ of the boundary, i.e. for a small positive constant d_0. Without loss of generality we can assume that the boundary datum $\psi(x)$ from (1.2) extends into Ω in a way that close to the boundary (i.e. in Ω_0) it is constant along normals. Since d_0 can be chosen sufficiently small and $\partial\Omega$ is assumed to be of class C^2, for every $x \in \Omega_0$ there exists a unique closest point $y(x)$ on $\partial\Omega$. We shall then denote the interior unit normal to $\partial\Omega$ at $y(x)$ by $\nu(x)$.

The following condition provides Lipschitz continuity on the boundary of continuous viscosity solutions u to the Dirichlet problem (1.1)(1.2).

There exist nonnegative constants M and β and a function φ satisfying (2.2) such that

$$F(x, \psi(x), D\psi(x) - t\nu, 0)\ \operatorname{sign}(t) \le d^\beta(x)\ \varphi^{1+\beta}(|t|)\ |t|^{2+\beta} \tag{2.4}$$

for every $x \in \overline{\Omega}_0$ and $|t| \ge M$.

Theorem 2.3. (Global boundary gradient estimate for $\beta \ge 0$)
Suppose that (1.5)–(1.7) and (2.4) hold and that ψ and $\partial\Omega$ are of class C^2. Then every viscosity solution $u \in C(\overline{\Omega})$ of the Dirichlet problem (1.1)(1.2) is Lipschitz continuous up to the boundary. More precisely, there exists a positive constant $b \le d_0$ and a nonnegative function $h \in C^2[0, b]$ with $h(0) = 0$ and

$$|u(x) - \psi(x)| \le h(d(x)) \tag{2.5}$$

for every $x \in \overline{\Omega_b} := \{x \in \Omega;\ d(x) < b\}$, provided one of the following conditions holds:

i) $\beta = 0$ in (2.4), or

ii) $\beta > 0$ in (2.4), $\partial\Omega$ is convex and $\partial\Omega$ is strictly mean-convex with respect to the interior unit normal to $\partial\Omega$, or

iii) $\beta > 0$ in (2.4), $\partial\Omega$ is convex and there is a nontrivial subset γ of $\partial\Omega$ where $\partial\Omega$ may be just convex and only $\partial\Omega \setminus \gamma$ is strictly mean-convex with respect to the interior unit normal to $\partial\Omega$, but then

$$D^2\psi(x) \le 0 \quad \text{for every } x \in N_\gamma \cap \overline{\Omega} \text{ in a neigbourhood } N_\gamma \text{ of } \gamma. \tag{2.6}$$

Let us recall that in a canonical coordinate system with x_n pointing in normal direction into Ω, the boundary can be parametrized as $x_n = g(x_1, \ldots, x_{n-1})$. Then $\partial\Omega$ is called convex at $x \in \partial\Omega$ if the Hessian matrix $D^2 g(x) \geq 0$ and mean convex if $trace[D^2 g(x)] \geq 0$. This means that all principal curvatures are nonnegative (for convex surfaces) and that their sum is positive (for mean convex surfaces).

Notice that by the triangle inequality and by (2.5) we have $|u(x) - u(z)| \leq |u(x) - u(y(x))| + |u(y(x)) - u(z)|$ for $x \in \Omega_b$ and $z \in \partial\Omega$. If $y(x)$ denotes the closest point to x on $\partial\Omega$, one can conlude that $|u(x) - u(z)| \leq h(d(x)) + |\psi(z) - \psi(y(x))| \leq K_1|x - z|$. Therefore u is locally Lipschitz continuous on $\partial\Omega$. As an immediate consequence of (2.5) we obtain

Corollary 2.4.
Under the assumptions of Theorem 2.3 every viscosity solution of the viscosity Dirichlet problem (1.1)(1.2) satisfies the Dirichlet condition in a classical sense.

The statement of Corollary 2.4 follows immediately from estimate (2.5).

In the case that (2.4) is not satisfied everywhere in $\overline{\Omega_0}$, Theorem 2.3 does not apply. Nevertheless we can still prove local Lipschitz regularity and local estimates like (2.5) at every point $z \in \partial\Omega$ if we replace (2.4) by the slightly stronger condition, that there exist a function φ satisfying (2.2) and nonnegative constants M and β that may depend on z, such that

$$|F(x, \psi(x), p, 0)| \leq d^\beta(x)\ \varphi^{1+\beta}(|p|)\ |p|^{2+\beta} \tag{2.7}$$

for every $x \in \overline{B(z, R)} \cap \overline{\Omega} \subset \overline{\Omega_0}$ and $|p| \geq M$.

Theorem 2.5. (Local boundary gradient estimate for $\beta \geq 0$)
Suppose (1.5)–(1.7) and (2.7) hold and ψ and $\partial\Omega$ are of class C^2.

Then every viscosity solution $u(x) \in C(\overline{\Omega})$ of the Dirichlet problem (1.1) (1.2) is Lipschitz continuous on $\partial\Omega \cap \overline{B(z, R/2)}$, and the estimate (2.5) is satisfied for every $x \in \overline{B(z, R/2)} \cap \overline{\Omega}$ and some nonnegative function $h \in C^2[0, b]$ with $h(0) = 0$, provided at least one of the following conditions i) – iii) holds:

i) $\beta = 0$ in (2.7), or

ii) $\beta > 0$ in (2.7) and $\overline{B(z, R)} \cap \partial\Omega$ is convex and strictly mean convex with respect to the interior unit normal ν to $\partial\Omega$; or

iii) $\beta > 0$ in (2.7) and $\overline{B(z, R)} \cap \partial\Omega$ is convex with respect to ν but not strictly mean convex and the inequality (2.6) is satisfied for every $x \in \overline{B(z, R)} \cap \overline{\Omega}$.

In the following theorem we will show that the assumptions in Theorem 2.3 and 2.5 are sharp for the Lipschitz regularity of the continuous solutions of (1.1), (1.2). For this purpose let us formulate the supergrowth conditions for the lower order term $F(x, r, p, 0)$ for large values of $|p|$, which are complementary to conditions

(2.4), (2.7) in Theorems 2.3 and 2.5, and lead to boundary gradient blow up of the viscosity solutions when the boundary data $\psi(x)$ have large enough gradient.

Suppose that for some $z \in \partial\Omega$ either the estimates (2.8) or (2.9) hold for every $r \in \mathbb{R}$, $t \geq M$, $|p| \geq M$ and for some nonnegative constants M, β and some function ϕ satisfying (2.3)

$$\begin{cases} (i) & F(x, r, t\nu, 0) \leq -d^{\beta}(x)\phi^{1+\beta}(t)\, t^{2+\beta} \text{ for } x \in \overline{B(z,R)} \cap \overline{\Omega}, \\ (ii) & F(x, r, p, 0) \leq 0 \text{ for } x \in \Omega, \end{cases} \tag{2.8}$$

$$\begin{cases} (i) & F(x, r, -t\nu, 0) \geq d^{\beta}(x)\phi^{1+\beta}(t)\, t^{2+\beta} \text{ for } x \in \overline{B(z,R)} \cap \overline{\Omega}, \\ (ii) & F(x, r, p, 0) \geq 0 \text{ for } x \in \Omega, \end{cases} \tag{2.9}$$

Here $\overline{B(z,R)} \cap \overline{\Omega}$ is a one-sided neighborhhod of z. Conditions (2.8) *(ii)* and (2.9) *(ii)* will be used to coerce the solution into developing a large slope when the datum ψ has large variations near $z \in \Omega$.

Theorem 2.6. (Local gradient blow up on the boundary for $\beta \geq 0$)
Suppose (1.5)–(1.7) hold and $\partial\Omega \in C^2$. Moreover, assume that for some $z \in \partial\Omega$ condition (2.8) is satisfied with

$$\beta = 0 \qquad \text{or} \tag{2.10}$$

$$\beta > 0 \text{ and } B(z,R) \cap \partial\Omega \text{ is concave wrt the interior normal } \nu \text{ to } \partial\Omega. \tag{2.11}$$

Then there exist data $\psi_0(x) \in C^\infty(\overline{\Omega})$, $\psi_0(z) = 0$, $\psi_0(x) \geq 0$ such that every viscosity solution $u(x) \in C(\overline{\Omega})$ of the Dirichlet problem (1.1) with data $\psi_0(x)$ has infinite positive derivative $\frac{\partial u}{\partial \nu}$ in at least one point $z^* \in \partial\Omega \cap B(z, \varepsilon)$ in the sense that

$$\limsup_{t \to 0+} \frac{u(z^* + t\nu(z^*)) - u(z^*)}{t} = \infty.$$

Analogously, if (2.9) holds as well as (2.10) or (2.11), then every viscosity solution $u(x) \in C(\overline{\Omega})$ of the Dirichlet problem (1.1), (1.2) with data $-\psi_0(x)$ has infinite negative derivative $\frac{\partial u}{\partial \nu}$ in at least one point $z_* \in \partial\Omega \cap B(z, \varepsilon)$ in the sense that

$$\liminf_{t \to 0+} \frac{u(z^* + t\nu(z^*)) - u(z^*)}{t} = -\infty.$$

Theorem 2.6 shows that for larger choices of the boundary datum $|\psi| > |\psi_0|$ a viscosity solution u must fail to satisfy the Dirichlet condition in the classical sense. Instead it satisfies the boundary condition in the viscosity sense, i.e. the differential equation holds in boundary points in viscosity sense.

The assumptions of Theorem 2.6 can be simplified if $F(x, r, p, X)$ satisfies the strict monotonicity condition (1.4). Then part *(ii)* of the assumptions (2.8) and (2.9) become superfluous because (1.4) guarantees large slope of the viscosity solutions of (1.1) when the data have the same properties.

Corollary 2.7.

Suppose (1.4) - (1.6) hold and $\partial\Omega \in C^2$. Moreover, assume that for some $z \in \partial\Omega$ condition (2.8) (i) is satisfied as well as (2.10) or (2.12)

Then there exist data $\psi_0(x) \in C^\infty(\partial\Omega)$, $\psi_0(z) = 0$, $\psi_0(x) \geq 0$ such that every viscosity solution $u(x) \in C(\overline{\Omega})$ of the Dirichlet problem (1.1) with data $\psi_0(x)$ has infinite positive derivative $\frac{\partial u}{\partial \nu}$ in at least one point $z^* \in \partial\Omega \cap B(z, \varepsilon)$.

Analogously, if (2.9) (i) holds as well as (2.10) or (2.12), then every viscosity solution $u(x) \in C(\overline{\Omega})$ of the Dirichlet problem (1.1), (1.2) with data $-\psi_0(x) \in C^\infty(\overline{\Omega})$ has infinite negative derivative $\frac{\partial u}{\partial \nu}$ in at least one point $z_* \in \partial\Omega \cap B(z, \varepsilon)$.

While the previous theorems make statements about the gradient on the boundary $\partial\Omega$, the following results describe our investigations on the behaviour of the gradient in the interior of the domain. It turns out that the one–dimensional case is considerably easier to study than the multi–dmensional case. Therefore we first present our results in one dimension. In the one–dimensional situation the local interior gradient estimates for the continuous viscosity solutions of (1.1) hold essentially under the same growth conditions for the lower order term $F(x, r, p, 0)$ as in the case of the local boundary gradient estimates in Theorem 2.5. More precisely, we assume that for some $z \in \Omega$ there exist nonnegative constants K_0, β and M and a function $\varphi(t)$ satisfying (2.2) such that

$$|F(x, r, p, 0)| \leq |x - z|^\beta \varphi^{1+\beta}(|p|)\, |p|^{2+\beta} \tag{2.12}$$

holds for every $x \in B(z, R) \cap \Omega$ and every $|r| \leq K_0$ and $p \in \mathbb{R}^n$, $|p| \geq M$.

Theorem 2.8. (Interior gradient estimates for $\beta \geq 0$, $n = 1$)
Suppose that $\Omega = (-l, l)$, $\beta \geq 0$ and that (1.5)–(1.7) and (2.12) hold. Then every viscosity solution $u(x) \in C(-l, l)$ of (1.1) with $|u(x)| \leq K_0$ for $x \in (-l, l)$, is Lipschitz continuous at $z \in (-l, l)$ and the estimate

$$|u(x) - u(z)| \leq C_3 |x - z| \tag{2.13}$$

is true for every $x \in (-l, l)$, with a constant C_3 depending only on M, K_0, β, φ and $\operatorname{dist}(z, \pm l)$.

In order to show the sharpness of the conditions in Theorem 2.8 we have to construct "interior boundaries" for viscosity solutions of (1.1) (1.2) under suitable choice of the boundary data. Without loss of generality, we place the singulaity at $0 \in (-l, l)$. Let us first list our results in the case $\beta > 0$.

For this purpose we assume that there exist constants M, $\beta > 0$ and a function ϕ satisfying (2.3) such that

$$\operatorname{sign}(x)\; F(x, r, t, 0) \leq -|x|^\beta \phi^{1+\beta}(t) t^{2+\beta} \qquad \text{in } (-l, l), \tag{2.14}$$

for $r \in \mathbb{R}$, $t \geq M$

Theorem 2.9. (Interior gradient blow up for $\beta > 0$, $n = 1$)
Suppose (1.5)–(1.7) and (2.14) hold, and that $\beta > 0$ *and* $\Omega = (-l, l)$. *Then there exists a critical value* $M_* > 0$ *such that a viscosity solution* $u \in C(\overline{\Omega})$ *of the Dirichlet problem (1.1) (1.2) with Dirichlet datum* $\psi_1(\pm l) = \pm M_*$ *will have infinite gradient at the origin. Moreover,* $u(0) = 0$.

Remark 2.10. *From the proof of Theorem 2.9 it follows that there are no continuous viscosity solutions when the Dirichlet data* ψ *satisfy* $(-1)^i\psi((-1)^i l) > M_*$ *fir* $i = 1, 2$. *Instead the solutions will be discontinuous at* 0, *but they have (different) continous extensions from* $(-l, 0)$ *and* $(0, l)$ *to* 0. *Since there is no satisfactory theory of discontinuos viscosity solutions, while there is a theory that can accomodate detachment of viscosity solutions from Dirichlet data on the boundary, we like to think of* 0 *in this case as a new boundary and of the restrictions of* u *to* $(-l, 0)$ *and* $(0, l)$ *as viscosity solutions to separate Dirichlet problems in* $(-l, 0)$ *and* $(0, l)$ *with vanishing Dirichlet data at* 0. *In this context we speak of* 0 *as an* **interior boundary**. *Roughly speaking, transition layers are then interpreted as boundary layers. An example of such behaviour is given by Example 1 in [46].*

In the case $\beta = 0$, condition (2.14) contradicts the continuity of F, because $F(x, r, -t\nu, 0)$ would have to be strictly positive in $(-l, 0)$ and simultaneously strictly negative in $(0, l)$. If u is monotone increasing in $(-l.l)$ then an interior gradient blow up requires that it has an inflection point where it switches from convex to concave. Therefore we cannot expect any interior gradient blow up in the case $\beta = 0$, $n = 1$. In fact, if F happens to be independent of x, one can find a proof of interior Lipschitz regularity for $\beta = 0$ and general $n \geq 1$ in Theorem 5 of [45].

Let us now present our results on the multi–dimensional case. For $\beta = 0$, conditions (2.12) are the so–called natural growth conditions for Lipschitz regularity, see [31, 37, 50, 54, 70]. In view of the one–dimensional situation it seems natural to conjecture that analogues of Theorems 2.8 and 2.9 hold for general $\beta \geq 0$. However, a priori estimates in higher dimensions are much more delicate. A counterexample of Safonov [58] seems to suggest that we can only expect such estimates (without imposing additional structural assumptions on F) in the autonomous case where $F(u, Du, D^2u)$ does not depend on x. In what follows we assume

$$F(x, u, p, X) \text{ in (1.1) is independent of } x \in \Omega. \tag{2.15}$$

Theorem 2.11. (Global Lipschitz estimates for $\beta \geq 0$, $n \geq 1$)
Suppose that (1.5)–(1.7) and (2.4), (2.15) hold. Then every viscosity solution $u(x) \in C(\Omega)$ *of (1.1) is Lipschitz continuous at* $z \in \Omega$ *and the estimate*

$$|u(x) - u(z)| \leq C_4|x - z| \tag{2.16}$$

is true for every $x, z \in \overline{\Omega}$, *with a constant* C_4 *depending only on* M, φ β, Ω *and boundary data* ψ.

Using Theorem 2.9 one can construct situations in higher dimensions, where the gradient of u blows up in the interior along a flat level surface of u for $\beta > 0$ and $n > 1$. Suppose for simplicity that $\Omega \subset \mathbb{R}^n$ is Steiner-symmetric, i.e. convex in direction x_n and reflection symmetric about the plane $\{x_n = 0\}$. We define $\Omega_+ := \Omega \cap \{x_n > 0\}$ and $\Omega_- := \Omega \cap \{x_n < 0\}$ and $E := \Omega \cap \{x_n = 0\}$. If there exist constants M and $\beta > 0$ and a function ϕ satisfying (2.3) such that

$$F(x, r, te_n, 0) \geq |x_n|^\beta \phi^{1+\beta}(t) t^{2+\beta} \quad \text{in } \Omega_-, \tag{2.17}$$

$$F(x, r, -te_n, 0) \leq -|x_n|^\beta \phi^{1+\beta}(t) t^{2+\beta} \quad \text{in } \Omega_+, \tag{2.18}$$

for $r \in \mathbb{R}$, $t \geq M$ and $e_n = (0, \ldots, 0, 1) \in \mathbb{R}^n$, then $\overline{E}$ can become an "iinterior boundary" under suitably chosen Dirichlet data.

Corollary 2.12. (Interior gradient blow up for $\beta > 0$, $n > 1$)
Suppose (1.5)–(1.7) and (2.17), (2.18) hold, and that $\beta > 0$ and Ω is Steiner-symmetric in x_n and has boundary $\partial\Omega$ of class C^2. Then there exists critical data $\psi_*(x)$ such that a viscosity solution $u \in C(\overline{\Omega})$ of the Dirichlet problem (1.1) (1.2) with Dirichlet datum $\psi_*(x) = h^*(x_n)$ has infinite slope in direction x_n along E. Here h^* depends only on x_n and is defined in the proof. Moreover, $u(x) = 0$ on E.

Our proof shows that Steiner symmetry of Ω is not essential. In fact similar examples can be constructed for any bounded domain of class C^2 and any intersecting hyperplane.

Finally let us remind the reader that for $\beta = 0$ we do not expect interior gradient blow up, because, as explained above, at least for $n = 1$ blow up can only occur on the boundary.

The following examples illustrate the sharpness of our results when $\beta > 0$.

Example 2.13. *Consider*

$$-u''(x) - \operatorname{sign}(x)|x|^\beta \varphi^{1+\beta}(|u'(x)|)\ |u'(x)|^{2+\beta} = 0, \ \textit{in } (-b, b) \tag{2.19}$$
$$u(-b) = -B, \ u(b) = B,$$

where b, B, β are positive constants and $\varphi(t)$ satisfies (2.2). Problem (2.19) has a unique classical and odd solution $u(x) \in C^2[-b, b]$, given by $u(x) = h(x)$ for $x \in [0, b]$, and $u(x) = -h(-x)$ for $x \in [-b, 0)$. For positive t the function $h(t)$ is explicitly defined in Corollary 4.2 as a solution of equation (4.1) in $(0, b)$ with boundary data $h(0) = 0$, $h(b) = B$. For negative t it can be continued as an odd function.

Example 2.14. *In contrast, consider now*

$$-u''(x) - \operatorname{sign}(x)|x|^\beta \phi^{1+\beta}(|u'(x)|)\ |u'(x)|^{2+\beta} = 0 \ \textit{in } (-b, b), \tag{2.20}$$
$$u(-b) = -B, \ u(b) = B,$$

where b, B, β are positive constants and $\phi(t)$ satisfies (2.3). For every $b > 0$ there exists a critical value K^*, depending on b, β and ϕ (see Corollary 4.4) such that for all $B < K^*$ problem (2.20) has a unique classical and odd solution $u(x) \in C^2[-b, b]$

$$u(x) = \begin{cases} h(x) & \text{for} \quad x \in [0, b] \\ -h(-x) & \text{for} \quad x \in [-b, 0) \end{cases} \tag{2.21}$$

where the function $h(x) \in C^2[0, b]$ is defined in Corollary 4.4 as a solution of equation (4.6) with $h(0) = 0$, $h(b) = B$. When $B = K^*$, then (2.20) has a unique continuous viscosity solution h^* given by (2.21) with the function $h^*(x)$, $h^*(0) = 0$, $h^*(b) = K^*$ defined in (4.8)(4.9). The solution is classical except at the origin, i.e. $u \in C^2([-b, b] \setminus \{0\}) \cap C[-b, b]$, but $u(x)$ is not Lipschitz continuous at zero because $u'(0) = \infty$. Finally, when $B > K^*$, then (2.20) has no continuous viscosity solution.

A piecewise solution can be constructed via the Perron method [18] and is given by $u(x) = h^*(x) + B - K^*$ for $x \in [0, b]$, $u(x) = -h^*(-x) - B + K^*$ for $x \in [-b, 0]$. Observe that now $u \in C^2([-b, b] \setminus \{0\}) \cap C[0, b] \cap C[-b, 0]$ and that u is jump discontinuous at the origin.

There are similar examples for $\beta = 0$ on the half-interval $(0, b)$ with $u(0) = 0$ and $u(b) = B$. For $\phi(t) = t^{q-2}$ and $q > 2$ the critical boundary value K^* was explicitly calculated in [1].

3 Discussion of Results

The assumptions for the boundary and interior Lipschitz regularity of the continuous viscosity solutions of (1.1), (1.2) in Section 2 are optimal and sharp, as Example 2.14 and our blow up results Theorem 2.6, Corollary 2.7, Theorem 2.9 and Corollary 2.12 show. If the assumptions for Lipschitz regularity fail, then there exist smooth Dirichlet data $\psi(x)$ with large enough gradient such that either (1.1), (1.2) has no viscosity solution in $C(\overline{\Omega})$, i.e. Perron's solutions are discontinuous, or the corresponding viscosity solution is not Lipschitz continuous in the interior or on the boundary of Ω. Our assumptions depend on the growth of the lower order term $F(x, r, p, 0)$ for large values of $|p|$, which can be so strong that the solution can become discontinuous. In that sense it then resembles a nonlinear hyperbolic equation. At the same time the strongly elliptic principal term has a smoothing effect on the solutions. The competition between these two effects is reflected in conditions (2.7) and (2.8), (2.9) in the sense that the smoothing effect wins and viscosity solutions have bounded gradients when (2.7) holds with φ satisfying (2.2), whereas the blow up effect wins and viscosity solutions have unbounded gradient when (2.8), (2.9) holds with ϕ satisfying (2.3).

In some cases, when $\beta > 0$ in (2.4) or (2.7), the conditions for boundary gradient estimates in Theorem 2.3 and 2.5 depend on the geometry of the boundary

$\partial\Omega$ and the choice of the boundary data. As far as we can tell, this observation appears to be new for uniformly elliptic equations. For $\beta = 0$ there is no such restriction on the geometry of $\partial\Omega$.

As for the interior Lipschitz regularity, we conjecture that the interior gradient estimates remain true under the general condition (2.12) with $\beta \geq 0$. However, our proof cannot be extended to this general case because we have no information on the geometry of the level line of the solution through some interior point of Ω. In fact, interior gradient estimates were proved in [37], [6], [7], [68] with different methods and under different additional structure assumptions on F. Let us explicitly point out that our idea for the proof of the interior and boundary estimates differs from [37], [6], [7], [68], because we use only classical barrier functions and a very weak formulation of the comparison principle, Theorem 3 in [43], in which one of the viscosity sub- or supersolutions is a classical one, whereas the method of proof in [37], [6], [7] is based on the idea of doubling variables and follows [18]. Other gradient estimates which are mentioned in the introduction were obtained by interpolation from $C^{1,\alpha}$ estimates. Therefore they provide only sufficient, but not necessary conditions for gradient bounds.

Note also that the boundary estimates in Theorem 2.3 and 2.5 give not only the Lipschitz regularity of the continuous viscosity solutions but also shed light on the problem when the Dirichlet condition is satisfied in classical sense and when it is only satisfied in the viscosity sense. In the latter case the solution may detach from the given Dirichlet data and instead solves the differential equation on some part of the boundary.

For more details on the notion of boundary conditions in classical or viscosity sense we recommend Section 7 in [18]. It is curious to mention that even classical solutions of the equation can violate boundary conditions if the equation is rather degenerate (see Example 7.8 in [18]). In Proposition 7.11 in [18] one can find a sufficient condition under which smooth sub- or supersolutions in $C^2(\bar{\Omega})$ satisfy the boundary data in classical sense. The result in [18] is extended in [34] to continuous viscosity sub- or supersolutions in $C(\bar{\Omega})$ of fully nonlinear degenerate elliptic equations under the "natural growth" conditions. Roughly speaking, for uniformly elliptic equations "natural growth" means quadratic growth in particuar of the lower order term $F(x, r, p, 0)$ (but also F_x, F_u etc.) with respect to the gradient variables p. In fact, the result in [34] is sharp in the power scale of the nonlinearities, but in the refined power-log scale of the nonlinearities our results in Theorem 2.3 and 2.5 in the present paper are slightly more general than those in [34] and are the best possible ones (see Example 1 in [46]). Moreover, we show that not only the nonlinearity of the equation, but also the geometry of the domain and the choice of the boundary data is crucial for the validity of the classical Dirichlet condition.

Necessary conditions for the violation of the classical Dirichlet problem are the assumptions in Theorem 2.6 and Corollary 2.7 that guarantee boundary gradient blow-up. For the time being, the boundary and interior gradient blow up phenomena are not well understood even for classical solutions. The results in [1, 2, 3, 4, 12, 13, 21, 25, 26, 30, 45, 53, 55, 57, 63, 64, 66, 67] are basically applied to classical solutions of quasilinear parabolic equations whose first space derivatives blow up after finite time. While the blow up on the boundary leads to detachment of the solution from the data and to solutions of the viscosity Dirichlet problem (1.1), 1.2), the interior gradient blow up produces solutions that are discontinuous in the interior of Ω. Unfortunately, the theory of discontinuous viscosity solutions is still not developed. Only indirectly, after a suitable regularization of the equation and after passing to the viscosity limit, or by the Perron method or numerically [55] one can see the resulting discontinuous solutions. Coincidentally, the conditions for boundary and interior gradient estimates in Section 2 are almost identical in the one–dimensional case. That is why our main conjecture is that viscosity solutions, as well as classical ones, have gradient blow up only at the "boundary". Here "boundary" means not the topological one but that part of the boundary of the corresponding boundary value problem, where the data should be prescribed. From the notion of viscosity solutions to the Dirichlet problem (1.1) (1.2), Definition 2.2, it is clear that a part of the topological boundary may be free from boundary data and these points are "interior" points in the sense that the equation is satisfied rather than the data. A simple example of Fichera

$$\begin{aligned} -y^2u_{xx} + 2xyu_{xy} - x^2u_{yy} + 2xu_x + 2yu_y &= 0 \qquad \text{in } B(0,1) \\ u &= \psi(x,y) \text{ on } \partial B(0,1) \end{aligned} \tag{3.1}$$

illustrates that the unit sphere $\partial B(0,1)$ in $\mathbb{R}^2$ is irrelevant for boundary data. At the same time elementary calculations show that prescribing u at the center of the ball determines u uniquely in the entire ball. In this example the origin can be interpreted as an "interior boundary" of boundary value problem (3.1). In fact, in terms of polar coordinates (3.1) can be rewritten as a parabolic equation

$$ru_r - u_{\theta\theta} = 0 \quad \text{for } (r,\theta) \in (0,1) \times (0,2\pi)$$

with periodic boundary conditions on $\theta = 0$ and $\theta = 2\pi$ and initial data at $r = 0$, i.e. at the center of the disc. By the way, the notion of the "boundary" as a beginning of some diffusion process for degenerate elliptic equations is also perceived and maybe better understood in the context of probability theory (see [27]). However, the question when and where the viscosity solutions form "interior boundaries" or equivalently have "an interior boundary- or transition layer" is still essentially unanswered. It is clear from [45] that for autonomous strictly elliptic fully nonlinear equations with locally Lipschitz coefficients in the sense of (1.6) the interior blow up phenomenon does not occur. We show in Example 2.14 explicit viscosity solutions which have interior gradient blow up at zero.

In Theorem 2.9 we construct "interior boundaries" in the one–dimensional case, choosing suitable Dirichlet data. At the same time there is no gradient blow up of the viscosity solutions on the boundary. This follows from simple concavity-convexity properties of the solutions near the boundary.

4 Boundary gradient estimates and boundary blow up

We start this section with some auxiliary results for special nonlinear ordinary differential equations. In the PDE-setting, suitable barrier functions will satisfy similar equations along normals to the boundary. Therefore the investigation of the ODEs is essential for the proofs of our results in the multidimensional case, i.e. Theorems 2.3, 2.5, 2.6 and Corollary 2.7, but also Theorems 2.8 and 2.9. Moreover, these ODEs illustrate our results in the one-dimensional case by providing explicit solutions that display the essential features of gradient blow up.

A fully nonlinear ordinary differential equation $F(t, u, u', u'') = 0$ that is strongly elliptic in the sense of (1.5) can always be transformed into explicit form $-u'' - f(t, u, u') = 0$. For our purposes it will suffice to study lower order terms $f(t, u, u')$ which are independent of u and which are the product of a power of t and a function of u'. In this case we can explicitly integrate the ordinary differential equation. Let us remark in passing that gradient blow up in the autonomous case $-\varepsilon u'' - f(u) = 0$ has been treated to some extent in [44].

Lemma 4.1.
For all constants $m, K, \beta \geq 0, b_0 > 0$ and function $\varphi(t)$ satisfying (2.2), there exists a positive constant $b = b(m, K, \beta, b_0, \varphi) \leq b_0$ and a nonnegative function $h \in C^2[0, b]$ such that

$$\begin{aligned} &-h''(t) - t^\beta \varphi^{1+\beta}(|h'(t)|)|h'(t)|^{2+\beta} = 0 \ \textit{in} \ (0, b) \\ &h(0) = 0, \quad h(b) \geq K, \quad \textit{and} \quad h'(t) \geq m \ \textit{in} \ [0, b] \end{aligned} \tag{4.1}$$

Proof. We choose positive constants $L > L_1 \geq m$ such that

$$\int_{L_1}^{L} \frac{ds}{s\varphi(s)} \geq K, \ \int_{L_1}^{L} \frac{ds}{s^{2+\beta}\varphi^{1+\beta}(s)} < \frac{b_0^{\beta+1}}{\beta + 1}$$

from the divergence of the first integral and the convergence of the second one.

Setting $b := \left[(1 + \beta) \int_{L_1}^{L} \frac{ds}{s^{2+\beta}\varphi^{1+\beta}(s)}\right]^{1/(1+\beta)} \leq b_0$ and

$$H(z) := \int_{z}^{L} \frac{ds}{s^{2+\beta}\varphi^{1+\beta}(s)}, \ H(z) : [L_1, L] \to [0, b]$$

the function $h(t) := \int_0^t H^{-1}(s^{1+\beta}/(1 + \beta))ds$ turns out to be the desired solution. In fact

$$h'(t) = H^{-1}(t^{1+\beta}/(1 + \beta)) \geq L_1 \geq m, \ h(0) = 0$$

and

$$\begin{aligned} h(b) &= \int_0^b H^{-1}\left(\frac{s^{1+\beta}}{1+\beta}\right) ds \\ &= -\int_L^{L_1} \frac{t\,dt}{t^{2+\beta}\varphi^{1+\beta}(t)[(1+\beta)H(t)]^{\beta/(1+\beta)}} \end{aligned}$$

after the change $s^{1+\beta} = (1+\beta)H(t)$. Simple computations give us from the monotonicity of $\varphi(s)$ the following chain of inequalities

$$\begin{aligned} h(b) &\geq (1+\beta)^{-\frac{\beta}{1+\beta}} \int_{L_1}^{L} \frac{dt}{t^{1+\beta}\varphi(t)[\int_t^L ds/s^{2+\beta}]^{\frac{\beta}{1+\beta}}} \\ &= L^\beta \int_{L_1}^{L} \frac{dt}{t\varphi(t)(L^{1+\beta}-t^{1+\beta})^{\beta/(1+\beta}} \geq \int_{L_1}^{L} \frac{dt}{t\varphi(t)} \geq K. \end{aligned}$$

Corollary 4.2.
For any $b > 0$, $K \geq 0$, $\beta \geq 0$ and φ satisfying (2.2), the two point boundary value problem

$$-h''(t) - t^\beta \varphi^{1+\beta}(|h'(t)|)\ |h'(t)|^{2+\beta} = 0 \text{ in } (0,b) \tag{4.2}$$
$$h(0) = 0 \qquad \text{and} \qquad h(b) = K \tag{4.3}$$

has a unique classical solution $h \in C^2([0,b])$.

In fact, this solution is explicitly constructed in the proof of Lemma 4.1

Lemma 4.3.
For all constants $m, \beta \geq 0$, $b_0 > 0$ and function $\phi(t)$ satisfying (2.3), there exist a positive constant $b = b(m,\beta,b_0,\phi) \leq b_0$ and a nonnegative function $h \in C^2(0,b] \cap C[0,b]$ such that

$$\begin{aligned} &-h''(t) - t^\beta \phi^{1+\beta}(|h'(t)|)\,|h'(t)|^{2+\beta} = 0 \qquad \text{in } (0,b), \\ &h(0) = 0, \quad h'(t) \geq m, \quad \text{and} \quad h'(0) = \infty. \end{aligned} \tag{4.4}$$

Proof. We choose a positive constant $L \geq m$ sufficiently large such that

$$\int_L^\infty \frac{ds}{s^{2+\beta}\phi^{1+\beta}(s)} \leq \frac{b_0^{1+\beta}}{1+\beta}$$

from the convergence of the integral and $b := \left[(1+\beta)\int_L^\infty \frac{ds}{s^{2+\beta}\phi^{1+\beta}(s)}\right]^{1/(1+\beta)} \leq b_0$.

The function $h(t) = \int_0^t H^{-1}\left(\frac{s^{1+\beta}}{1+\beta}\right) ds$, where

$$H(z) = \int_z^\infty \frac{ds}{s^{2+\beta}\phi^{1+\beta}(s)},\, H(z) : [L,\infty] \to [0,b]$$

has the desired properties. We will check only the boundedness of $h(t)$. After the change of the variables $s^{1+\beta} = (1+\beta)H(\tau)$ simple computations give us the chain of inequalities

$$
\begin{aligned}
h(b) &= -(1+\beta)^{\frac{1}{1+\beta}} \int_L^\infty \tau d(H^{\frac{1}{1+\beta}}) \\
&= -(1+\beta)^{\frac{1}{1+\beta}} \tau \left(\int_\tau^\infty \frac{ds}{s^{2+\beta}\phi^{1+\beta}(s)} \right)^{\frac{1}{1+\beta}} \Bigg|_L^\infty \\
&+ (1+\beta)^{\frac{1}{1+\beta}} \int_L^\infty \left(\int_\tau^\infty \frac{ds}{s^{2+\beta}\phi^{1+\beta}(s)} \right)^{\frac{1}{1+\beta}} d\tau \\
&\le (1+\beta)^{\frac{1}{1+\beta}} L \left(\int_L^\infty \frac{ds}{s^{2+\beta}\phi^{1+\beta}(s)} \right)^{\frac{1}{1+\beta}} \\
&+ (1+\beta)^{\frac{1}{1+\beta}} \int_L^\infty \left(\int_\tau^\infty \frac{ds}{s^2\phi(s)} \right)^{\frac{1}{1+\beta}} \frac{d\tau}{(\tau\phi(\tau))^{\beta/(1+\beta)}} \\
&\le (1+\beta)^{\frac{1}{1+\beta}} L \left(\int_L^\infty \frac{ds}{s^{2+\beta}\phi^{1+\beta}(s)} \right)^{\frac{1}{1+\beta}} \\
&+ (1+\beta)^{\frac{1}{1+\beta}} \left(\int_L^\infty \frac{d\tau}{\tau\phi(\tau)} \right)^{\frac{\beta}{1+\beta}} \left(\int_L^\infty \left(\int_\tau^\infty \frac{ds}{s^2\phi(s)} \right) d\tau \right)^{\frac{1}{1+\beta}} \\
&= (1+\beta)^{\frac{1}{1+\beta}} \left[L \left(\int_L^\infty \frac{ds}{s^{2+\beta}\phi^{1+\beta}(s)} \right)^{\frac{1}{1+\beta}} + \int_L^\infty \frac{d\tau}{\tau\phi(\tau)} \right] \\
&\le (1+\beta)^{\frac{1}{1+\beta}} \left[\left(\int_m^\infty \frac{ds}{s\phi^{1+\beta}(s)} \right)^{\frac{1}{1+\beta}} + \int_m^\infty \frac{ds}{s\phi(s)} \right] =: K_1
\end{aligned}
$$

i.e. from the monotonicity of $h(t)$

$$
h(t) \le h(b) =: K^* \le K_1 \quad \text{for } t \in [0,b]. \tag{4.5}
$$

In the above estimates we used the fact that $\lim_{\tau\to\infty} \tau^{1+\beta} \int_\tau^\infty \frac{ds}{s^{2+\beta}\phi^{1+\beta}(s)} = 0$, that ϕ is monotone, so that $s^\beta\phi^\beta(s)$ can be estimated from below and pulled out of the double integral, the Cauchy-Schwarz inequality, and in the last equality we changed the order of integration. Moreover we used the fact that $m < L < s$ in the last inequality.

Corollary 4.4.

For any $b > 0$ and $K \ge 0$ the two point boundary problem

$$
-h''(t) - t^\beta \phi^{1+\beta}(|h'(t)|)\, |h'(t)|^{2+\beta} = 0 \textit{ in } (0,b) \tag{4.6}
$$

$$
h(0) = 0 \qquad \textit{and} \qquad h(b) = K \tag{4.7}
$$

has a unique classical solution $h \in C([0,b]) \cap C^2((0,b])$ if and only if $K \le K^*(b)$, where $K^*(b)$ is the value of $h^*(b)$ and h^* solves the initial value problem

$$
-h^{*\prime\prime}(t) - t^\beta \phi^{1+\beta}(|h^{*\prime}(t)|)\, |h^{*\prime}(t)|^{2+\beta} = 0 \textit{ in } (0,b) \tag{4.8}
$$

$$
h^*(0) = 0 \qquad \textit{and} \qquad h^{*\prime}(0) = \infty. \tag{4.9}
$$

Moreover, for $K < K^*(b)$ we have $h'(0) < \infty$ and $h \in C^2[0, b]$.

This follows from the explicit construction of h in the proof of Lemma 4.3.

Proof of Theorem 2.3. We want to apply Lemma 4.1 and choose m larger than M, 1, and the following quantities that depend on assumptions i) through iii). In case i) m must also exceed

$$\frac{c_2}{\varphi(1)}\left(n\sup_{x\in\partial\Omega}|k_i(x)| + \sup_{x\in\overline{\Omega}}\left|\operatorname{trace}\left[D^2\psi(x)\right]\right|\right),$$

in case ii), respectively iii), it should also be larger than

$$\frac{c_2}{c_1}\frac{\sup\limits_{x\in\overline{\Omega}}|\operatorname{trace}\,[D^2\psi(x)]\,|}{\inf\limits_{x\in\overline{\Omega}}\sum\limits_{i=1}^{n-1}k_i(x)} \quad \text{resp.} \quad \frac{c_2}{c_1}\frac{\sup\limits_{x\in\overline{\Omega}}|\operatorname{trace}\,[D^2\psi(x)]\,|}{\inf\limits_{x\in\partial\Omega\setminus N\gamma}\sum\limits_{i=1}^{n-1}k_i(x)}\,.$$

Here $k_i(x)$ are the principal curvatures of $\partial\Omega$. Lemma 4.1 also calls for $K = \sup_{x\in\overline{\Omega}}(|u(x)| + |\psi(x)|)$, β and b_0 as well as for a function φ satisfying (2.2). Some φ is already defined through condition (2.4), but it still satisfies (2.2) if it is replaced by $(2/c_1)^{\frac{1}{1+\beta}}\varphi$.

So according to Lemma 4.1 there exists a constant $0 < b \le b_0$ and a nonnegative function $h(t) \in C^2[0, b]$ satisfying

$$-h''(t) - \frac{2}{c_1}t^\beta\varphi^{1+\beta}(|h'(t)|)|h'(t)|^{2+\beta} = 0 \text{ in } (0, b) \tag{4.10}$$
$$h(0) = 0, \quad h(b) \ge \sup_{x\in\overline{\Omega}}(|u(x)| + |\psi(x)|) = K, \quad \text{and} \quad h'(t) \ge m \text{ in } [0, b]$$

We will show that the function

$$v(x) = \psi(x) + h(d(x)),\ v(x) \in C^2(\Omega_b) \cap C^0(\bar{\Omega}_b),\ \Omega_b = \{x \in \Omega;\ d(x) < b\}$$

is a classical supersolution of (1.1), (1.2).

It is clear that $v(x) = \psi(x)$ on $\partial\Omega$ and $v(x) = h(b) + \psi(x) \ge u(x)$ on $\partial\Omega \cap \partial\Omega_b$ from the choice of $h(t)$. Moreover, from (1.5), (1.7), (2.4) and (2.6) simple computations give us the following chain of inequalities.

$$\begin{aligned}
& F(x,v(x),Dv(x),D^2v(x))) \qquad (4.11)\\
= \; & F(x,\psi(x)+h(d),D\psi(x)+h'(d)\,\nu,D^2\psi(x)+h'(d)D^2d+h''(d)\nu\otimes\nu)\\
\geq \; & F(x,\psi,D\psi+h'\nu,D^2\psi+h'D^2d-\frac{2}{c_1}d^\beta(x)\varphi^{1+\beta}(h')\,(h')^{2+\beta}\,\nu\otimes\nu)\\
\geq \; & 2d^\beta\varphi^{1+\beta}(h')(h')^{2+\beta}+F(x,\psi,D\psi+h'\nu,0)\\
+ \; & c_1h'\sum_{i=1}^{n-1}(k_i)_+ + c_2h'\sum_{i=1}^{n-1}(k_i)_- - c_1\operatorname{trace}\{(D^2\psi)_-\} - c_2\operatorname{trace}\{(D^2\psi)_+\}\\
\geq \; & d^\beta\varphi^{1+\beta}(h')(h')^{2+\beta}+c_1h'\sum_{i=1}^{n-1}(k_i)_+ - c_2h'\sum_{i=1}^{n-1}|(k_i)_-| + c_1|\operatorname{trace}\{(D^2\psi)_-)\}|\\
& -c_2\operatorname{trace}\{(D^2\psi)_+\}\geq 0
\end{aligned}$$

from the choice of h. Here $(k_i)_+ = \max(k_i,0), (k_i)_- = \min(k_i,0), \{D^2\psi\} = \{(D^2\psi)_+\}+\{(D^2\psi)_-\}, \{(D^2\psi)_+\}$ is a nonnegative matrix, $\{(D^2\psi)_-\}$ is a nonpositive one. Moreover, the following identities were used in a principal coordinate system of the unit inner normal $\nu(x)$ to $\partial\Omega$ and the principal directions λ^i, $i=1,2,\dots,n-1$ of $\partial\Omega$ at the point $y(x)\in\partial\Omega$ nearest to $x\in\Omega_b$,

$$\begin{aligned}
\left\{D^2d(x)\right\} &= -\operatorname{diag}\left\{\frac{k_1}{1-k_1d},\dots,\frac{k_{n-1}}{1-k_{n-1},d},0\right\}\\
\frac{k_i}{1-k_id} &= k_i+\frac{k_i^2d}{1-k_id}\quad .
\end{aligned}$$

(see [31] or [59]).

If $u(x)\in C(\overline{\Omega})$ is a viscosity solution of the Dirichlet problem for (1.1), which happens to satisfy the boundary condition $u(x)=\psi(x)$ on $\partial\Omega$ in the classical sense, then from the comparison principle, Theorem 3 in [43], for viscosity sub- and supersolutions such that at least one of them is $C^2(\Omega)\cap C(\overline{\Omega})$ smooth we get the right inequality of (2.5). Note that the assumptions of Theorem 3 in [43] are satisfied with Lipschitz modulus of ellipticity and continuity of F $w_1(s)=w_2(s)=s$.

However, we are considering viscosity solutions $u(x)\in C(\overline{\Omega})$ that satisfy the Dirichlet condition only in the viscosity sense (see Definition 2.2), and so we need a more elaborate proof. Therefore we consider the following boundary value problem

$$\begin{aligned}
& f(x,w,Dw,D^2w) := \qquad (4.12)\\
& \quad = F(x,v(x)+w,Dv(x)+Dw,D^2v(x)+D^2w)=0 \;\text{ in } \Omega_b,\\
& w=u(x)-v(x)\leq 0 \;\text{ on } \partial\Omega_b\cap\Omega,\\
& w=0 \qquad\qquad\quad\;\text{ on } \partial\Omega_b\cap\partial\Omega. \qquad .
\end{aligned}$$

It is clear that f satisfies conditions (1.5), (1.6), (1.7), and from (4.11) we have that $f(x,0,0,0)=F(x,v(x),Dv(x),D^2v(x))\geq 0$ in Ω_b. Moreover, $w(x)=u(x)-$

$v(x)$ is a viscosity subsolution of (4.12) in Ω_b. In fact, from Remark 2.7 (ii) in [18] it follows that $\mathcal{J}^{2,+}_{\Omega_b} w(x) = (p - Dv(x), X - D^2 v(x))$ with $(p, X) \in \mathcal{J}^{2,+}_{\Omega_b} u(x)$ so that $f(x, w(x), p, X) = F(x, u(x), p, X) \leq 0$ according to Definition 2.1. From the strong interior maximum principle, Theorem 1 in [43], if $w(x)$ attains its positive maximum at some interior point $x_0 \in \Omega_b$, then $w(x) \equiv w(x_0) > 0$ for every $x \in \overline{\Omega_b}$ which is impossible on $\partial\Omega_b \cap \Omega$.

If $w(x)$ attains its positive maximum at some boundary point $x_0 \in \partial\Omega_b \cap \partial\Omega$ then $(0,0) \in \mathcal{J}^{2,+}_{\bar{\Omega}_b} w(x_0)$. From Remark 2.7 in [18] it follows that

$$(\lambda\nu(x_0), \mu\nu(x_0) \otimes \nu(x_0)) \in \mathcal{J}^{2,+}_{\bar{\Omega}_b} w(x_0)$$

for every $\lambda > 0$ and $\mu \in \mathbb{R}$. Since $w(x_0) > 0$, according to Definition 2.2, $w(x)$, as a viscosity subsolution of the differential equation (4.12), satisfies the equation and not the Dirichlet data at x_0, i.e. $f(x_0, w(x_0), q, Y) \leq 0$ for every $(q, Y) \in \mathcal{J}^{2,+}_{\bar{\Omega}_b} w(x_0)$.

But for $q = \lambda\nu(x_0)$ and $Y = \mu\nu(x_0) \otimes \nu(x_0)$ with fixed $\lambda > 0$ and for a real number $\mu \to -\infty$ the above inequality is impossible because

$$0 \geq f(x_0, w(x_0), \lambda\nu(x_0), \mu\nu(x_0) \otimes \nu(x_0)) \geq f(x_0, w(x_0), \lambda\nu(x_0), 0) + c_1|\mu| \to \infty \quad .$$

Thus we proved that $w(x) = u(x) - v(x) \leq 0$ in Ω_b, i.e. $u(x) \leq \psi(x) + h(d(x))$ in Ω_b for every viscosity solution of the Dirichlet problem (1.1), (1.2).

Analogously one can prove that

$$u(x) - \psi(x) + h(d(x)) \geq 0 \quad \text{in } \Omega_b \quad .$$

Proof of Theorem 2.5. The proof of Theorem 2.5 is essentially the same as the proof of Theorem 2.3. The only difference is that we consider a new domain $G \supset \Omega$, with boundary ∂G of class C^2, which coincides with $\partial\Omega$ only in $\overline{B(z, R/2)}$.

If $\varrho(x) = \operatorname{dist}(x, \partial G)$, then $\Omega_b = \{x \in \Omega;\ \varrho(x) < b\} \subset B(z, R)$ when b is sufficiently small and $\varrho(x) \in C^2(\Omega_b)$. One observes that

$$\varrho(x) \geq d(x) = \operatorname{dist}(x, \partial\Omega) \quad \text{for all } x \in \Omega_b$$

and $\varrho(x) = d(x)$ when the point $y(x) \in \partial G$ nearest to x belongs to $B(z, R/2) \cap \partial\Omega$. Then the rest of the proof is the same as the proof of Theorem 2.3, using barrier functions $h(\varrho(x))$ depending on the distance $\varrho(x)$ to ∂G instead of the distance $d(x)$ to $\partial\Omega$.

Proof of Theorem 2.6. Let $B_0 = B(z_0, R_0)$ be the ball with center z_0 and radius R_0 so that $\overline{B_0} \cap \overline{\Omega} = z$. If $b_0 > 0$ is sufficiently small, then $\Omega_0 = \{x \in \Omega;\ |x - z_0| < R_0 + b_0\} \subset B(z, R)$, $d(x) \in C^2(\Omega_0)$ and the estimates

$$\sup_{\partial\Omega \cap B(z,R)} |k_i(x)| \leq \frac{1}{2b_0} \quad , \quad i = 1, 2, \ldots, n-1 \tag{4.13}$$

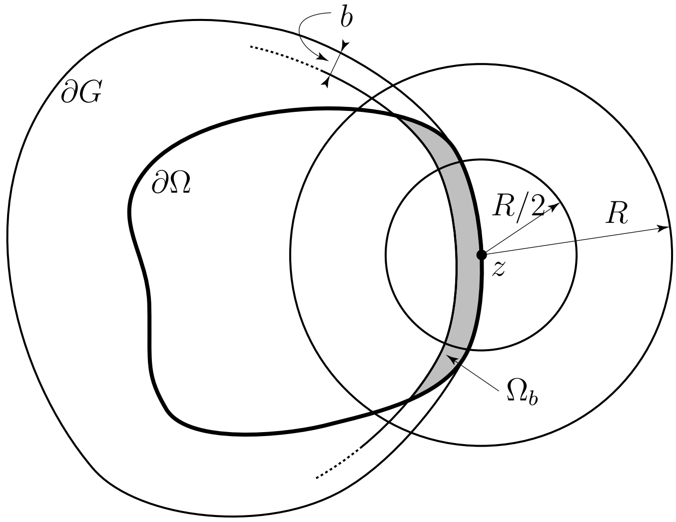

Figure 1: Proof of Theorem 2.5

hold, where k_i, $i = 1, 2, \ldots, n-1$, are the principal curvatures of $\partial\Omega$.

We want to apply Lemma 4.3 and choose the constants appropriately. We pick $m := \max\{M,\ 2nc_2/(b_0\phi(M))\}$, $\beta \geq 0$ and $b_0 > 0$ defined by (4.13). Arguing as in the proof of Theorem 2.3, we can replace the function ϕ that is given through (2.8)(2.9) by $\phi(t)/(2c_2)^{1/(1+\beta)}$. It will still satisfy (2.3). Hence there exist a positive constant $b = b(m, \beta, b_0, \phi, c_2) \leq b_0$ and a nonnegative function $h(t) \in C^2(0, b] \cap C[0, b]$ such that

$$\begin{aligned} &-h''(t) - \frac{1}{2c_2} t^\beta \phi^{1+\beta}(|h'(t)|)|h'(t)|^{2+\beta} = 0 \ \text{ in } (0, b) \\ &h(0) = 0,\ h'(0) = \infty,\ h'(t) \geq m \ . \end{aligned} \tag{4.14}$$

Moreover, as in the proof of Lemma 4.3, the estimate

$$h(t) \leq h(b) = K_* \leq [2c_2(1+\beta)]^{\frac{1}{1+\beta}} \left[\left(\int_M^\infty \frac{ds}{s\phi^{1+\beta}(s)} \right)^{\frac{1}{1+\beta}} + \int_M^\infty \frac{ds}{s\phi(s)} \right] \tag{4.15}$$

holds, where the constant factor that was added to ϕ has been accomodated.

Let us choose

$$\psi_0(x) = N(|x - z_0|^2 - R_0^2) \ , \quad \text{where } N = \frac{K_*}{2bR_0 + b^2} \ , \tag{4.16}$$

and note that $\psi_0(x) \in C^\infty(\overline{\Omega})$, $\psi_0(x) \geq 0$, $\psi_0(z) = 0$ and $\psi_0(x) = K_*$ for $|x - z_0| = R_0 + b$. Suppose that conditions (2.8) of Theorem 2.6 are satisfied. Since

$$\begin{aligned} & F(x, \psi_0(x), D\psi_0(x), D^2\psi_0(x)) \\ = \ & F(x, \psi_0(x), 2N(x - z_0), 2NI) \leq F(x, \psi_0(x), 2N(x - z_0), 0) - 2Nnc_1 \leq 0 \end{aligned}$$

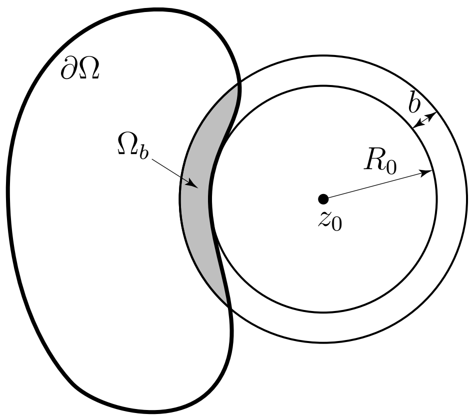

Figure 2: Proof of Theorem 2.6

in Ω it follows that $\psi_0(x)$ is a classical subsolution of (1.1), (1.2).

If $u(x) \in C(\overline{\Omega})$ is a viscosity solution of the Dirichlet problem (1.1) that happens to satisfy the Dirichlet condition (1.2) in the classical sense, then from the comparison principle, Theorem 3 in [43] we get

$$\psi_0(x) \leq u(x) \quad \text{in } \overline{\Omega} \quad . \tag{4.17}$$

If, however $u(x) \in C(\overline{\Omega})$ is a viscosity solution of the Dirichlet problem (1.1), (1.2) (see Definition 2.2) that satisfies the boundary condition only in the viscosity sense, we have to argue as follows. Suppose that $u(x) - \psi_0(x)$ has a negative minimum at some boundary point $x_0 \in \partial\Omega$, then $(0,0) \in \mathcal{J}^{2,-}_{\overline{\Omega}}(u(x_0) - \psi_0(x_0))$. From Remark 2.7 in [18] it follows that

$$(D\psi(x_0), D^2\psi_0(x_0)) \in \mathcal{J}^{2,-}_{\overline{\Omega}} u(x_0)$$

as well as

$$(D\psi_0(x_0) + \lambda\nu(x_0),\ D^2\psi_0(x_0) + \mu\nu(x_0) \otimes \nu(x_0)) \in \mathcal{J}^{2,-}_{\overline{\Omega}} u(x_0)$$

for fixed $\lambda < 0$ and arbitrary $\mu \in \mathbb{R}$.

Since $u(x_0) < \psi_0(x_0)$ then from Definition 2.2 $u(x)$ is a viscosity supersolution of (1.1) at x_0, i.e.,

$$F(x_0, u(x_0),\ D\psi_0(x_0) + \lambda\nu(x_0),\ D^2\psi(x_0) + \mu\nu(x_0) \otimes \nu(x_0) \geq 0$$

which is impossible for $\lambda < 0$ fixed and $\mu \to \infty$ because from (1.5)

$$\begin{aligned} 0 &\leq F(x_0, u(x_0), D\psi_0(x_0) + \lambda\nu(x_0), D^2\psi_0(x_0) + \mu\nu(x_0) \otimes \nu(x_0)) \\ &\leq F(x_0, u(x_0), D\psi_0(x_0) + \lambda\nu(x_0), D^2\psi_0(x_0)) - c_1\mu \to -\infty. \end{aligned}$$

Hence (4.17) holds for every viscosity solution $u(x) \in C(\overline{\Omega})$ of (1.1), (1.2).

We consider the function $h(d(x))$, h defined in (4.14), in the domain $\Omega_b = \{x \in \Omega;\ |x - z_0| < R_0 + b\}$. Simple computations give us from (1.5), (1.7), (2.8), (4.13) and (4.14) the following chain of inequalities, using the same notation as in the proof of Theorem 2.3

$$
\begin{aligned}
& F(x, h(d(x)), Dh(d), D^2h(d)) \qquad\qquad (4.18)\\
= \;& F(x, h, h'\nu, h''\nu \otimes \nu + h' D^2 d) \\
= \;& F\left(x, h, h'\nu, -\frac{1}{2c_2} d^\beta(x)\phi^{1+\beta}(h')(h')^{2+\beta}\nu \otimes \nu + h' D^2 d\right) \\
\leq \;& F(x, h, h'\nu, 0) \\
& + \frac{1}{2} d^\beta(x)\phi^{1+\beta}(h')(h')^{2+\beta} + c_2 h' \sum_{i=1}^{n-1} \frac{(k_i)_+}{1 - k_i d} + c_1 h' \sum_{i=1}^{n-1} \frac{(k_i)_-}{1 - k_i d} \\
\leq \;& \begin{cases} -\frac{1}{2}\phi(h')(h')^2 + b_0^{-1} n c_2 h' - c_1 h' \sum_{i=1}^{n-1} |(k_i)_-| \leq 0, & \text{if } \beta = 0. \\ -\frac{1}{2} d^\beta(x).\phi^{1+\beta}(h')(h')^{2+\beta} - c_1 h' \sum_{i=1}^{n-1} |(k_i)_-| \leq 0, & \text{if } \beta > 0. \end{cases}
\end{aligned}
$$

Notice that for $\beta > 0$ the terms involving $(k_i)_+$ are zero by assumption. Consequently $h(d(x))$ is a $C^2(\Omega_b) \cap C(\bar{\Omega}_b)$ smooth subsolution of equation (1.1) in Ω_b and from (4.14) – (4.17) we conclude that $h(d(x)) = 0 \leq \psi_0(x) \leq u(x)$ on $\partial\Omega \cap \partial\Omega_b$ and $h(d(x)) \leq K_* \leq \psi_0(x) \leq u(x)$ on $\Omega \cap \partial\Omega_b$.

Since $u(x) \geq \psi_0(x) \geq h(d(x))$ on all of $\partial\Omega_b$ then it follows from the comparison principle, Theorem 3 in [43], that $u(x) \geq h(d(x))$ in Ω_b. Now we have to distinguish two cases. When $u(z) = \psi_(z) = 0$, since $u(x) \geq h(d(x))$, $h(0) = 0$ and $h'(0)$ has an infinite positive gradient in the direction of the interior unit normal $\nu(z)$, the statement of Theorem 2.6 is proved. If u does not satisfy the Dirichlet condition in z in the classical sense, then $u(x) > \psi_0(x)$ in $D := \partial\Omega \cap B(z, \varepsilon)$. Therefore u is a viscosity subsolution of (1.1) in these points of detachment. Let us show that this can only happen if ∇u is unbounded, To see this we first show that $\mathcal{J}^{2,+}_{\overline{\Omega}} u(x) = \emptyset$ in D. Otherwise there exist a $y \in D$ such that $\mathcal{J}^{2,+}_{\overline{\Omega}} u(y) \neq \emptyset$, and therefore there exist $(q, Y) \in \mathcal{J}^{2,+}_{\overline{\Omega}} u(y)$. Then it follows from Remark 2.7 in [18] that also $(q + \lambda\nu(y), Y + \mu\nu(y) \otimes \nu(y)) \in \mathcal{J}^{2,+}_{\overline{\Omega}} u(y)$ for every $\lambda > 0$ and $\mu \in \mathbb{R}$. From Definition 2.1 and (1.5) we obtain for fixed $\lambda > 0$ and $\mu \to -\infty$ the following absurd chain of inequalities

$$
\begin{aligned}
0 \;&\geq\; F(y, u(y), q + \lambda\nu(y), Y + \mu\nu(y) \otimes \nu(y)) \\
&\geq\; F(y, u(y), q + \lambda\nu(y), Y) - \mu c_1 \to +\infty.
\end{aligned}
$$

Therefore $\mathcal{J}^{2,+}_{\overline{\Omega}} u(x) = \emptyset$ in D. But this is possible only when $u(x)$ has an infinite $\frac{\partial u}{\partial \nu}$ at some point z^* belonging to D. In fact, if u_ν were bounded everywhere in D, at some z^* one could construct an element (p, X) belonging to the (empty) set $\mathcal{J}^{2,+}_{\overline{\Omega}} u(z^*)$. Here is the construction of (p, X): First we find a function $g \in C^2(\partial\Omega)$ such that $g(x) \geq u(x)$ in D, and $g(z^*) = u(z^*)$ for some $z^* \in D$. Using the alleged boundedness of $u_\nu(x)$ from above, we can extend the function g as a C^2 function into Ω with large enough gradient, so that it stays above u in a neighbourhood of z^*. But then $p := Dg(z^*)$ and $X := D^2 g(z^*)$ belong to $\mathcal{J}^{2,+}_{\overline{\Omega}} u(z^*)$.

This completes the proof of Theorem 2.6 under assumption (2.8). The proof under assumption (2.9) is essentially the same, using similar estimates for $u(x)$ from above.

Proof of Corollary 2.7: The idea of the proof uses similar critical data ψ_0 as in (4.16), except that suitable constants N_1 are added or subtracted to make sure that $\psi_0 \mp N_1$ is a sub- or supersolution. Let us consider only the case (2.8)(i) in Corollary 2.7. The case (2.9)(i) is symmetric.

We will use the same notation as in the proof of Theorem 2.6 and we define essentially the same critical data $\psi_1(x) := \psi_0(x) - N_1 = N(|x - z_0|^2 - R_0^2) - N_1$ where ψ_0 and N are given in (4.16) and $N_1 > 0$ is chosen sufficiently large so that, observing (1.4),

$$F(x, \psi_0(x) - N_1, D\psi_0(x), D^2\psi_0(x)) \leq -c_0 N_1 + F(x, \psi_0(x), D\psi_0(x), D^2\psi_0(x)) \leq 0.$$

Hence, arguing as in the proof of (4.17), if $u(x) \in C(\overline{\Omega})$ is a viscosity solution of (1.1) with data $\psi_0(x) - N_1$ then from the comparison principle

$$\psi_0(x) - N_1 \leq u(x) \quad \text{in } \overline{\Omega} \quad .$$

The rest of the proof is identical with the proof of Theorem 2.6 using the barrier functions $h(d(x)) - N_1$ in the domain Ω_b.

5 Interior gradient estimates and interior gradient blow-up

5.1 The one-dimensional case

In this subsection we give first proofs of the regularity Theorem 2.8 and the blow up result Theorem 2.9. The idea of the proof of the interior gradient estimates is to compare the continuous viscosity solution of (1.1) with suitable barrier functions. The barrier functions are classical sub- and supersolutions of (1.1) so that the technique of the doubling the number of the independent variables as in [18] is not necessary. In fact only the strong interior maximum principle from [43] is used.

Proof of Theorem 2.8. Suppose $u(x) \in C(-l, l)$ is a viscosity solution of (1.1). Without loss of generality we assume that $z = 0$ and $u(0) = 0$. Thus we will prove Lipschitz continuity of $u(x)$ at the point 0 with a Lipschitz constant depending on c_i, $i = 1, 2$, φ, $\sup\limits_{x \in (-l/2, l/2)} |u(x)|, M$ and l.

We intend to apply Lemma 4.1 and set $m := \max\{M, 1, c_2/\varphi(1)\}$, $K = \sup\limits_{x \in (-l/2, l/2)} |u(x)|$, $\beta \geq 0$ and $b_0 = l/2$. Without loss of generality we replace

$\varphi(t)$ from condition (2.12) by $c_1^{-1/(1+\beta)}\varphi(t)$; then it still satisfies (2.2). So according to Lemma 4.1 there exists some $b \in (0, b_0]$ and a function $h(t) \in C^2[0, b]$ solving

$$h''(t) + \frac{t^\beta}{c_1}\left(\varphi(h'(t))^{1+\beta}\left(h'(t)\right)^{2+\beta} = 0 \quad \text{in } (0, b)$$

In the interval $I = (-b, b) \subset (-l, l)$ we consider its even extension $h_1(x) = h(|x|)$. Since $h \in C^2[0, b]$ and $h(0) = 0$, it is enough to prove the estimate

$$-h(|x|) \leq u(x) \leq h(|x|) \;\; \text{in } I. \tag{5.1}$$

If, for example, the right inequality in (5.1) fails, then the function $v(x) = u(x) - h(|x|)$ has a positive maximum at some point $y \in \overline{I}$, $y \neq 0$, i.e. $\max_{\overline{I}} v(x) = v(y) = M_1 > 0$. From the choice of $h(t)$ it follows that y is an interior point of I and $y \neq 0$. Without loss of generality suppose that $y \in (0, b)$, the other case is similar.

Therefore the function $v(x) = u(x) - h_1(x)$ has a positive maximum M_1 at the interior point $y \in (0, b)$. We will show that $v(x)$ is a viscosity subsolution of the equation

$$\begin{aligned} & f(x, v, Dv, D^2v) \\ = \;& F(x, v + h_1(x), Dv + Dh_1(x), D^2v + D^2h_1(x)) = 0 \;\; \text{in } (0, b). \end{aligned} \tag{5.2}$$

Since $u \in C([0, b])$ and $h_1 \in C^2([0, b])$, it follows from Remark 2.7 in [18] that for every $(p, X) \in \mathcal{J}^{2,+}_{[0,b]}v(x)$ we get $(p + Dh_1(x),\, X + D^2h_1(x)) \in \mathcal{J}^{2,+}_{[0,b]}u(x)$. Hence

$$f(x, v(x), p, X) = F(x, u(x), p + Dh_1(x), X + D^2h_1(x)) \leq 0$$

and according to Definition 2.1 $v(x)$ is a viscosity subsolution of equation (5.2).

Let us now check that $f(x, 0, 0, 0) \geq 0$ in $(0, b)$. In fact, this inequality holds because $h_1(x)$ is a classical supersolution of (1.1). From (1.5), (1.7), (2.2), (2.12) and the choice of $h(t)$ we get the following chain of inequalities.

$$\begin{aligned} & f(x, 0, 0, 0) = F(x, h_1, Dh_1, D^2h_1) \\ \geq \;& F\left(x, 0, h_1'(x), h_1''(x)\right) \\ = \;& F\left(x, 0, h_1'(x), -c_1^{-1}\,|x|^\beta\,\left(\varphi(h_1'(x))\right)^{1+\beta}\left((h_1'(x)\right)^{2+\beta}\right) \\ \geq \;& |x|^\beta\,\left(\varphi(h_1'(x))\right)^{1+\beta}\left((h_1'(x)\right)^{2+\beta} + F(x, 0, h_1'(x), 0) \geq 0 \;\;, \end{aligned}$$

Since all assumptions of the strong interior maximum principle, Theorem 1 in [43], are satisfied in $(0, b)$ we obtain that $v(x) \equiv M_1 > 0$ in $[0, b]$ which is impossible at the endpoints 0 and b.

Proof of Theorem 2.9. We will apply the second part of Corollary 4.4 and choose $b = l$, $\beta > 0$, and the function $\phi(t)/(c_2)^{\frac{1}{1+\beta}}$ satisfying (2.3). Then there exists a nonnegative function $h^*(t)$ with the properties (4.8) (4.9) satisfying

$$h^*(t) \leq h^*(l) = K^* \quad \text{in } [0, l]\ . \tag{5.3}$$

Let us call $M_* := K^*$ a critical boundary datum in Theorem 2.9. From (1.5), (1.7) and (2.14) it follows that h^* is a classical subsolution of (1.1) in $(0,l)$. In fact,

$$\begin{aligned} F(x,h^*,h^{*\prime},h^{*\prime\prime}) &= F\left(x,h^*,h^{*\prime},-c_2^{-1}x^\beta(\varphi(h^{*\prime}))^{1+\beta}\ \left(h^{*\prime}\right)^{2+\beta}\right) \\ &\leq x^\beta(\varphi(h^{*\prime}))^{1+\beta}\ \left(h^{*\prime}\right)^{2+\beta} + F(x,h^*,h^{*\prime},0)\ \leq\ 0\ . \end{aligned}$$

Let us assume that $u(x)$ is Lipschitz continuous at 0, i.e. there exists a constant C such that

$$|u(x)-u(0)| \leq C|x|\ . \tag{5.4}$$

We will show that $u(x) \geq h^*(x)$ under the assumption (5.4). If not, then $\sup\limits_{x\in[0,l]} v(x) = \sup\limits_{x\in[0,l]} (h^*(x)-u(x)) = v(x_0) > 0$ for some $x_0 \in [0,l]$. From the strong interior maximum principle, Theorem 1 in [43] we know that x_0 cannot lie in $(0,l)$, because $v(x)$ is a viscosity subsolution of (5.5) in $(0.l)$.

$$f(x,v,v',v'') = -F(x,h^*(x)) - v, h^{*\prime}(x) - v',\ h^{*\prime\prime}(x) - v'') = 0. \tag{5.5}$$

In fact, for every $(p,X) \in \mathcal{J}^{2,+}_{(0,l)}v(x)$ it follows that $(-p+h^{*\prime}(x), -X+h^{*\prime\prime}(x)) \in \mathcal{J}^{2,-}_{(0,l)}u(x)$ and hence

$$f(x,v(x),p,X) = -F(x,u(x),h^{*\prime}-p,h^{*\prime\prime}-X)\ \leq 0\ .$$

Moreover, $f(x,0,0,0) = -F(x,h^*,h^{*\prime},h^{*\prime\prime}) \geq 0$ and the conditions of Theorem 1 and 2 in [43] are satisfied.

If $v(x)$ attains a positive maximum at 0, then from the strong boundary maximum principle we get

$$\limsup_{x\to+0}\frac{v(x)-v(0)}{x} = \limsup_{x\to+0}\left[\frac{h^*(x)}{x} - \frac{u(x)-u(0)}{x}\right] < 0,$$

which contradicts (5.4) and the choice of $h^*(x)$, with $h^{*\prime}(0) = \infty$. On the other hand, if $v(x)$ attains a positive maximum at l, then $u(l) < M_*$ and $u(x)$ is a viscosity supersolution of (1.1) at l. Note that $(0,0) \in \mathcal{J}^{2,+}_{[0,l]}v(l)$ and h^* is of class C^2 near l, we can conclude from Remark 2.7 in [18] that $(h^{*\prime}(l), h^{*\prime\prime}(l)) \in \mathcal{J}^{2,-}_{[0,l]}u(l)$ as well as $(h^{*\prime}(l)+\lambda, h^{*\prime\prime}(l)+\mu) \in \mathcal{J}^{2,-}_{[0,l]}u(l)$ for any $\lambda < 0$ and any $\mu \in \mathbb{R}$. For negative and fixed λ and $\mu \to \infty$ we obtain from (1.5) the following contradiction.

$$0 \leq F(l,u(l),h^{*\prime}(l)+\lambda,h^{*\prime\prime}(l)+\mu) \leq F(l,u(l),h^{*\prime}(l)+\lambda,h^{*\prime\prime}(l)) - c_1\mu \to -\infty$$

Thus $v(x) \leq 0$ in $[0,l]$, i.e.

$$h^*(x) \leq u(x) \ \text{ in } [0,l] \tag{5.6}$$

and in particular $u(0) \geq 0$. In the same way one can prove the estimate

$$u(x) \leq -h^*(-x) \ \text{ in } [-l,0] \tag{5.7}$$

and in particular $u(0) \leq 0$. Combining (5.6) and (5.7) we get the continuity of u across 0. But $h'(0) = \infty$ and this contradicts our assumption that $u(x)$ is Lipschitz continuous at the origin.

5.2 The multi–dimensional case

Proof of Theorem 2.11. According to Theorem 2.3 the viscosity solution of (1.1) (1.2) is globally Lipschitz continuous on the boundary and satisfies (1.2) in the classical sense. This implies through (2.5) that there exists a constant C_5 such that

$$|u(x) - u(y)| \le C_5 \, |x - y| \quad \text{ for any } x \in \overline{\Omega} \text{ and } y \in \partial\Omega.$$

i.e. for $(x, y) \in \partial(\Omega \times \Omega)$. But now global Lipschitz continuity follows from Theorem 5 in [45] or (alternatively) from Theorem VII.1 in [37]. Incidentally, assumption (7.4) in the local regularity result Theorem VII.2 of [37], has been weakened to condition (3.12) in [7], but neither one of those needs to hold for our result on the autonomous situation.

The **proof of Corollary 2.12** follows the same ideas as the proof of Theorem 2.9. The critical datum $\psi_*(x)$ depends only on one variable x_n and $h^*(x_n)$ is the odd extension of h^* from Corollary 4.4. Note that in this case the data ψ_* are not of class C^2 in $\partial\Omega \cap \{x_n = 0\}$. There they are merely continuous and have infinite gradient in tangential direction. Everywhere else on $\partial\Omega$ they are of class C^2.

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