A Liouville comparison principle for solutions of singular quasilinear elliptic second-order partial differential inequalities

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Abstract

We compare entire weak solutions u and v of quasilinear partial differential inequalities on $\mathbb{R}^n$ without any assumptions on their behaviour at infinity and show among other things, that they must coincide if they are ordered, i.e. if they satisfy $u \geq v$ in $\mathbb{R}^n$. For the particular case that $v \equiv 0$ we recover some known Liouville type results. Model cases for the equations involve the p-Laplacian operator for $p \in [1, 2]$ and the mean curvature operator.

1 Introduction and Definitions.

This work is devoted to the study of a Liouville comparison principle for entire weak solutions of quasilinear elliptic second-order differential inequalities of the forms

$$A(u) + |u|^{q-1}u \leq A(v) + |v|^{q-1}v \tag{1}$$

and

$$-A(u) + |u|^{q-1}u \leq -A(v) + |v|^{q-1}v \tag{2}$$

on $\mathbb{R}^n$, where $n \in \mathbb{N}$, $0 < q \in \mathbb{R}$, and the operator $A(w)$ belongs to the class of so-called α-monotone operators. Typical examples of such operators are the p-Laplacian operator

$$\Delta_p(w) := \operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right) \tag{3}$$

for $1 \le p \le 2$, its well-known modification (see, e.g. p.155 in [17])

$$\widetilde{\Delta}_p(w) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial w}{\partial x_i}\right|^{p-2}\frac{\partial w}{\partial x_i}\right) \tag{4}$$

for $1 \le p \le 2$, and the mean curvature operator

$$M(w) := \operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right). \tag{5}$$

Remark 1 *Note that if u and v satisfy the inequalities*

$$-A(u) \ge |u|^{q-1}u \qquad \text{and} \qquad -A(v) \le |v|^{q-1}v, \tag{6}$$

or

$$A(u) \ge |u|^{q-1}u \qquad \text{and} \qquad A(v) \le |v|^{q-1}v, \tag{7}$$

then the pair (u,v) satisfies inequality (1) or (2). Thus, all the results obtained in this paper for solutions of (1) or (2) are valid for the corresponding solutions of system (6) or (7).

To be specific, let $A(w)$ be the differential operator in divergence form

$$A(w) = \sum_{i=1}^{n} \frac{d}{dx_i} A_i\left(x, \nabla w\right), \tag{8}$$

and assume that the functions $A_i\left(x,\xi\right)$, $i = 1,\dots,n$, satisfy the Carathéodory conditions on $\mathbb{R}^n \times \mathbb{R}^n$: They are continuous in ξ for almost every $x \in \mathbb{R}^n$ and measurable in x for all $\xi \in \mathbb{R}^n$.

Definition 1 *Let $n \in \mathbb{N}$ and $\alpha \ge 1$. The operator $A(w)$ given by (8) is called α-monotone iff $A_i\left(x,0\right) = 0$ for $i = 1,\dots,n$ and for a.e. $x \in \mathbb{R}^n$, if it is monotone in the sense that*

$$0 \le \sum_{i=1}^{n}\left(\xi_i^1 - \xi_i^2\right)\left(A_i\left(x,\xi^1\right) - A_i\left(x,\xi^2\right)\right), \tag{9}$$

and if there exists a positive constant $\mathcal{K}$ such that

$$\left(\sum_{i=1}^{n}\left(A_i\left(x,\xi^1\right) - A_i\left(x,\xi^2\right)\right)^2\right)^{\alpha/2} \le \mathcal{K}\left(\sum_{i=1}^{n}\left(\xi_i^1 - \xi_i^2\right)\left(A_i\left(x,\xi^1\right) - A_i\left(x,\xi^2\right)\right)\right)^{\alpha-1} \tag{10}$$

for every pair $\xi^1, \xi^2 \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$.

Note that condition (9) is the well-known monotonicity condition in PDE theory, while condition (10) is the proper α-monotonicity condition for differential operators, considered first in [12], see also [14]. In the particular case $\xi^2 = 0$ condition (10) implies the growth condition $|\overrightarrow{A}| \leq \mathcal{K}|\overrightarrow{\xi}|^{\alpha-1}$.
The notion of α-monotonicity was inspired by condition (3.7) in [18]. For the case $\xi^2 = 0$ the α-monotonicity condition boils down to the corresponding condition of Miklyukov with $A(x,u,Du) = A(x,Du)$ independent of u.
It is well-known that the mean curvature operator $M(w)$, the 1-Laplacian operator $\Delta_1(w)$ and its modification $\widetilde{\Delta}_1(w)$ satisfy condition (9) for $\alpha = 1$. Moreover it is easy to see that these three operators satisfy also condition (10) for $\alpha = 1$ with $\mathcal{K} = \sqrt{n}$.
Now we present algebraic inequalities which imply immediately that the p-Laplacian operator $\Delta_p(w)$ and its modification $\widetilde{\Delta}_p(w)$ satisfy the α-monotonicity condition for $\alpha = p$ and $1 < p \leq 2$.

Lemma 1 *Suppose $n \in \mathbb{N}$ and $1 < \alpha \leq 2$, and $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are vectors in $\mathbb{R}^n$ of length $|a|$ and $|b|$. Then there exists a positive constant $\mathcal{K}$ such that the inequalities*

$$\left(\sum_{i=1}^{n} \left(a_i|a|^{\alpha-2} - b_i|b|^{\alpha-2}\right)^2\right)^{\alpha/2} \leq \mathcal{K}\left(\sum_{i=1}^{n} (a_i - b_i)\left(a_i|a|^{\alpha-2} - b_i|b|^{\alpha-2}\right)\right)^{\alpha-1} \tag{11}$$

and

$$\left(\sum_{i=1}^{n} \left(a_i|a_i|^{\alpha-2} - b_i|b_i|^{\alpha-2}\right)^2\right)^{\alpha/2} \leq \mathcal{K}\left(\sum_{i=1}^{n} (a_i - b_i)\left(a_i|a_i|^{\alpha-2} - b_i|b_i|^{\alpha-2}\right)\right)^{\alpha-1} \tag{12}$$

hold.

A proof of this Lemma was given in [12], see also [15].

Note that there exist α-monotone differential operators with arbitrary degeneracy. For example, the weighted mean curvature operator

$$M^*(w) := \operatorname{div}\left(\frac{a(x)\nabla w}{\sqrt{1+|\nabla w|^2}}\right). \tag{13}$$

and the weighted p-Laplacian operator

$$\Delta_p^*(w) := \operatorname{div}\left(a\,(x)\,|\nabla w|^{p-2}\nabla w\right) \tag{14}$$

(see, e.g. p. 55 in [7]) with any measurable non-negative uniformly bounded weight-function $a(x)$ on $\mathbb{R}^n$ are α-monotone, with $\alpha = 1$ and $\alpha = p$ for any fixed $1 \le p \le 2$.

Now we can define weak solutions of (1) and (2).

Definition 2 *Suppose $\alpha \ge 1$, $q > 0$ and the operator $A(w)$ is α-monotone. We call a pair of functions (u, v) which belong to the space $W^{1,\alpha}_{\rm loc}(\mathbb{R}^n) \cap L^q_{\rm loc}(\mathbb{R}^n)$ an entire weak solution of (1) if*

$$\int\limits_{\mathbb{R}^n} \left[\sum_{i=1}^n \varphi_{x_i} A_i(x, \nabla u) - |u|^{q-1} u \varphi \right] dx \ge \int\limits_{\mathbb{R}^n} \left[\sum_{i=1}^n \varphi_{x_i} A_i(x, \nabla v) - |v|^{q-1} v \varphi \right] dx \tag{15}$$

for every non-negative test function $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Definition 3 *Similarly suppose that $\alpha \ge 1$, $q > 0$ and the operator $A(w)$ is α-monotone. Then we call a pair of functions (u, v) in $W^{1,\alpha}_{\rm loc}(\mathbb{R}^n) \cap L^\infty_{\rm loc}(\mathbb{R}^n)$ an entire weak solution of (2) if*

$$\int\limits_{\mathbb{R}^n} \left[-\sum_{i=1}^n \varphi_{x_i} A_i(x, \nabla u) - |u|^{q-1} u \varphi \right] dx \ge \int\limits_{\mathbb{R}^n} \left[-\sum_{i=1}^n \varphi_{x_i} A_i(x, \nabla v) - |v|^{q-1} v \varphi \right] dx \tag{16}$$

for every non-negative test function $\varphi \in C_0^\infty(\mathbb{R}^n)$.

We use analogous definitions for solutions to the systems (6), (7) or for the equality case in (2):

$$-A(u) + |u|^{q-1} u = -A(v) + |v|^{q-1} v. \tag{17}$$

2 Results.

In Theorems 1–5 we formulate our results for solutions of inequality (1) and in Theorem 6 for solutions of (2).

Theorem 1 *Let $n \ge 1$, $2 \ge \alpha \ge 1$, $\alpha \ge n$ and $q > 0$. Suppose that the operator $A(w)$ is α-monotone and (u, v) is an entire weak solution of inequality (1) on $\mathbb{R}^n$ such that $u(x) \ge v(x)$. Then $u(x) = v(x)$ on $\mathbb{R}^n$.*

Theorem 2 *Let $n \geq 2$, $2 \geq \alpha > 1$, $n > \alpha$, $\frac{n(\alpha-1)}{n-\alpha} \geq q > \alpha - 1$ and $q \geq 1$. Suppose that the operator $A(w)$ is α-monotone and (u, v) is an entire weak solution of inequality (1) on $\mathbb{R}^n$ such that $u(x) \geq v(x)$. Then $u(x) = v(x)$ on $\mathbb{R}^n$.*

Theorem 3 *Let $n \geq 2$, $2 \geq \alpha > 1$, $n > \alpha$, $q > \frac{n(\alpha-1)}{n-\alpha}$ and $q \geq 1$, and let the operator $A(w)$ be α-monotone. Then there exists no entire weak solution (u, v) of inequality (1) on $\mathbb{R}^n$ such that $u(x) \geq v(x)$ and the relation*

$$\limsup_{R\to+\infty} R^{-n+\frac{\alpha(q-\nu)}{q-\alpha+1}} \int\limits_{|x|<R} (u-v)^{q-\nu} dx = +\infty \tag{18}$$

holds for any $\nu \in (0, \alpha - 1]$.

To illustrate the sharpness of Theorem 3 we give Example 1.

Example 1 *For $n \geq 2$, $2 \geq \alpha > 1$, $n > \alpha$, $q > \frac{n(\alpha-1)}{n-\alpha}$, $q \geq 1$ and a suitable constant $c > 0$, the pair (u, v) of the functions*

$$u(x) = c(1 + |x|^{\alpha/(\alpha-1)})^{(1-\alpha)/(q-\alpha+1)} \qquad \text{and} \qquad v(x) \equiv 0 \tag{19}$$

is an entire smooth solution of inequality (1) on $\mathbb{R}^n$ with $A(w) = \Delta_\alpha(w)$ or $\mathcal{A}(w) = \widetilde{\Delta}_\alpha(w)$ satisfying $u(x) \geq v(x)$, while for any $\nu \in (0, \alpha - 1]$ (18) is violated because

$$0 < \limsup_{R\to+\infty} R^{-n+\frac{\alpha(q-\nu)}{q-\alpha+1}} \int\limits_{|x|<R} (u-v)^{q-\nu} dx < \infty. \tag{20}$$

The following statement is a simple corollary of Theorem 3.

Corollary 1 *Let $n \geq 2$, $2 \geq \alpha > 1$, $n > \alpha$, $q > \frac{n(\alpha-1)}{n-\alpha}$ and $q \geq 1$, and let the operator $A(w)$ be α-monotone. Then, for any given constants $c > 0$ and $\delta > 0$, there exists no entire weak solution (u, v) of inequality (1) on $\mathbb{R}^n$ such that $u(x) \geq v(x) + c(1 + |x|^{\alpha/(\alpha-1)})^{(1-\alpha)/(q-\alpha+1)+\delta}$.*

Theorem 4 *Let $n \geq 2$, $2 \geq \alpha \geq 1$, $n > \alpha$ and $1 \geq q > 0$, and let the operator $A(w)$ be α-monotone. Then there exists no entire weak solution (u, v) of inequality (1) on $\mathbb{R}^n$ such that $u(x) \geq v(x)$ and*

$$\limsup_{R\to+\infty} R^{\alpha-n} \int\limits_{\{|x|<R\}\cap\{x:\, u(x)\neq v(x)\}} (|u|^{q-1}u - |v|^{q-1}v)(u-v)^{1-\alpha} dx = +\infty. \tag{21}$$

To illustrate the sharpness of Theorem 4 we give three examples.

Example 2 *For $n \geq 2$, $2 \geq \alpha > 1$, $n > \alpha$, $\alpha - 1 > q > 0$, $0 < \mu < \frac{n-\alpha}{n}$, $\lambda = (\alpha-1)/(\alpha-1-q)$ and a suitable constant $c > 0$ the pair (u, v) of functions given by*

$$u(x) = c(1 + |x|^{\alpha/(\alpha-1)})^{\lambda} + (1 + |x|^{\alpha/(\alpha-1)})^{-\mu} \tag{22}$$

and

$$v(x) = c(1 + |x|^{\alpha/(\alpha-1)})^{\lambda} \tag{23}$$

is an entire smooth solution of inequality (1) on $\mathbb{R}^n$ with $A(w) = \Delta_\alpha(w)$ or $A(w) = \widetilde{\Delta}_\alpha(w)$ such that $u(x) \geq v(x)$ and

$$0 < \limsup_{R\to+\infty} R^{\alpha-n} \int\limits_{\{|x|<R\}\cap\{x:\, u(x)\neq v(x)\}} (|u|^{q-1}u - |v|^{q-1}v)(u-v)^{1-\alpha} dx < \infty. \tag{24}$$

Example 3 *For $n \geq 2$, $2 \geq \alpha > 1$, $n > \alpha$, $\alpha - 1 > q > 0$, $0 < \mu < \frac{n-\alpha}{n}$, $\lambda > (\alpha-1)/(\alpha-1-q)$, and a suitable constant $c > 0$, the pair (u, v) of functions (22) and (23) is an entire smooth solution of inequality (1) on $\mathbb{R}^n$ with $A(w) = \Delta_\alpha(w)$ or $A(w) = \widetilde{\Delta}_\alpha(w)$ such that $u(x) \geq v(x)$ and*

$$\limsup_{R\to+\infty} R^{\alpha-n} \int\limits_{\{|x|<R\}\cap\{x:\, u(x)\neq v(x)\}} (|u|^{q-1}u - |v|^{q-1}v)(u-v)^{1-\alpha} dx = 0. \tag{25}$$

Example 4 *To illustrate the sharpness of Theorem 4 in the case when $1 \geq q > 0$ and $q > \frac{n(\alpha-1)}{n-\alpha}$, we note that for $n \geq 2$, $2 \geq \alpha > 1$, $n > \alpha$, $1 \geq q > 0$, $q > \frac{n(\alpha-1)}{n-\alpha}$, and a suitable constant $c > 0$, the pair (u, v) of functions described in (19) is an entire smooth solution of (1) on $\mathbb{R}^n$ with $A(w) = \Delta_\alpha(w)$ or $A(w) = \widetilde{\Delta}_\alpha(w)$ such that $u(x) \geq v(x)$ and*

$$0 < \limsup_{R\to+\infty} R^{\alpha-n} \int\limits_{\{|x|<R\}\cap\{x:\, u(x)\neq v(x)\}} (|u|^{q-1}u - |v|^{q-1}v)(u-v)^{1-\alpha} dx < \infty. \tag{26}$$

The following statement follows immediately from Theorem 4 and supplements the results of Theorem 1 for $n > \alpha$ and Theorem 2 for $q = \alpha - 1$.

Theorem 5 *If $n \geq 3$, $\alpha = 2$ and $q = 1$ and if the operator $A(w)$ is α-monotone, then there exists no entire weak solution (u, v) of (1) on $\mathbb{R}^n$ such that $u(x) > v(x)$.*

Now we formulate our results for solutions of inequality (2).

Theorem 6 *Let $n \geq 1$, $2 \geq \alpha \geq 1$, $q \geq 1$ and $q > \alpha - 1$. Let the operator $A(w)$ be α-monotone, and let (u, v) be an entire weak solution of inequality (2) on $\mathbb{R}^n$. Then $u(x) \leq v(x)$ on $\mathbb{R}^n$.*

Corollary 2 *Let $n \geq 1$, $2 \geq \alpha \geq 1$, $q \geq 1$ and $q > \alpha - 1$. Let the operator $A(w)$ be α-monotone, and let (u, v) be an entire weak solution of the equation case of (2) on $\mathbb{R}^n$. Then $u(x) = v(x)$ on $\mathbb{R}^n$.*

To illustrate the sharpness of Theorem 6 and Corollary 2 we give some examples.

Example 5 *For $n \geq 1$ and $q = \alpha - 1 = 1$, the pair (u, v) of the functions*

$$u(x_1, \ldots, x_n) = \exp(x_1) \qquad \textit{and} \qquad v(x) \equiv 0 \tag{27}$$

is an entire smooth solution of equation (17) on $\mathbb{R}^n$ with $A(w) = \Delta(w)$, the Laplacian operator, such that $u(x) > v(x)$.

Example 6 *For $n \geq 1$, $\alpha \geq 1$ and $q = \alpha - 1$, the pair (u, v) of functions given in (27) is an entire smooth solution of equation*

$$-A(u) + (\alpha - 1)|u|^{q-1}u = -A(v) + (\alpha - 1)|v|^{q-1}v \tag{28}$$

on $\mathbb{R}^n$ with $A(w) = \Delta_\alpha(w)$ or $A(w) = \widetilde{\Delta}_\alpha(w)$ such that $u(x) > v(x)$.

Example 7 *For $n \geq 1$, $2 \geq \alpha > 1$, $\alpha - 1 > q > 0$, and a suitable constant $c > 0$, the pair (u, v) of functions*

$$u(x) = c|x|^{\frac{\alpha}{\alpha - 1 - q}} \qquad \textit{and} \qquad v(x) \equiv 0 \tag{29}$$

is an entire smooth solution of the equation (17) on $\mathbb{R}^n$ with $A(w) = \Delta_\alpha(w)$ or $A(w) = \widetilde{\Delta}_\alpha(w)$ such that $u(x) > v(x)$.

Remark 2 *In the special case when $u \equiv 0$ or $v \equiv 0$, the results of this paper are commonly called Liouville type theorems. In fact for inequalities of the form (1) the corresponding Liouville theorems were obtained in [12], [16] and [19], and they were inspired by a similar result for solutions of semilinear equations in [6]. For inequalities of the form (2) the corresponding Liouville theorems were obtained in [11] and [13], and they were inspired by a similar result for solutions of semilinear equations in [2]. Various interesting generalizations of those results can be found in [21] and [5]. In fact there is an abundance of literature on Liouville theorems, see for instance [1, 3, 4, 19, 20].*

In contrast to all of these, in the present paper we compare a pair (u, v) of nonzero solutions to (1) or (2). That is why we call this type of result Liouville comparison theorem. To prove these results we further develop the approach that was proposed to solve similar problems in wide classes of partial differential equations and inequalities in [10] and [12].

3 Proofs

In what follows, a "smooth" function is a C^∞-function on $\mathbb{R}^n$, and $B(R)$ is the open ball on $\mathbb{R}^n$ centered at the origin with radius $R > 0$.

Proof of Theorem 1. By assumption

$$\int\limits_{\mathbb{R}^n} \sum_{i=1}^n (A_i(x, \nabla u) - A_i(x, \nabla v))\varphi_{x_i} dx \geq \int\limits_{\mathbb{R}^n} (|u|^{q-1}u - |v|^{q-1}v)\varphi dx \qquad (30)$$

holds for every non-negative test function $\varphi \in C_0(\mathbb{R}^n)$. Let R and ε be arbitrary positive numbers, and let $\zeta : \mathbb{R}^n \to [0, 1]$ be a smooth cut-off function which equals 1 on $\overline{B(R/2)}$, 0 outside $B(R)$ and satisfies

$$|\nabla \zeta(x)| \leq \frac{c_0}{R} \qquad (31)$$

on $\mathbb{R}^n$. Without loss of generality, we substitute $\varphi(x) = (w(x) + \varepsilon)^{-\nu}\zeta^s(x)$ as a test function in (30), where $w(x) = u(x) - v(x)$, and where ν and s are real numbers such that $\nu > \alpha - 1$ and $s \geq \alpha$. Integrating by parts in (30) gives

$$-\nu \int\limits_{B(R)} \sum_{i=1}^n (A_i(x, \nabla u) - A_i(x, \nabla v)) w_{x_i} (w + \varepsilon)^{-\nu-1} \zeta^s dx$$

$$+s\int\limits_{B(R)}\sum_{i=1}^{n}(A_i(x,\nabla u)-A_i(x,\nabla v))\zeta_{x_i}(w+\varepsilon)^{-\nu}\zeta^{s-1}dx$$

$$:= \quad I_1+I_2\geq\int\limits_{B(R)}(|u|^{q-1}u-|v|^{q-1}v)(w+\varepsilon)^{-\nu}\zeta^{s}dx. \tag{32}$$

Observing that I_1 is non-positive by (9), we first estimate I_2 in terms of I_1. Since

$$|I_2| \quad = \quad \left|s\int_{B(R)}\sum_{i=1}^{n}(A_i(x,\nabla u)-A_i(x,\nabla v))\zeta_{x_i}(w+\varepsilon)^{-\nu}\zeta^{s-1}dx\right|$$

$$\leq \quad s\int_{B(R)}\left[\sum_{i=1}^{n}(A_i(x,\nabla u)-A_i(x,\nabla v))^2\right]^{1/2}|\nabla\zeta|(w+\varepsilon)^{-\nu}\zeta^{s-1}dx, \tag{33}$$

using condition (10) on the coefficients of the operator A, we easily obtain the following estimate from (33)

$|I_2|\leq$

$$\leq\int\limits_{B(R)}s\mathcal{K}^{\frac{1}{\alpha}}\left(\sum_{i=1}^{n}w_{x_i}(A_i(x,\nabla u)-A_i(x,\nabla v))\right)^{\frac{\alpha-1}{\alpha}}|\nabla\zeta|(w+\varepsilon)^{-\nu}\zeta^{s-1}dx. \tag{34}$$

We estimate the integrand on the right-hand side of (34) using Young's inequality

$$\mathcal{AB}\leq\rho\mathcal{A}^{\frac{\beta}{\beta-1}}+\rho^{1-\beta}\mathcal{B}^{\beta} \tag{35}$$

with $\rho=\frac{\nu}{2}$, $\beta=\alpha$,

$$\mathcal{A}=(w+\varepsilon)^{\frac{(1+\nu)(1-\alpha)}{\alpha}}\zeta^{\frac{s(\alpha-1)}{\alpha}}\left(\sum_{i=1}^{n}w_{x_i}(A_i(x,\nabla u)-A_i(x,\nabla v))\right)^{\frac{\alpha-1}{\alpha}}$$

and

$$\mathcal{B}=s\mathcal{K}^{1/\alpha}|\nabla\zeta|\zeta^{\frac{s}{\alpha}-1}(w+\varepsilon)^{\frac{\alpha-1-\nu}{\alpha}},$$

and arrive at

$$|I_2|\leq\frac{\nu}{2}\int\limits_{B(R)}\sum_{i=1}^{n}w_{x_i}(A_i(x,\nabla u)-A_i(x,\nabla v))(w+\varepsilon)^{-\nu-1}\zeta^{s}dx$$

$$+\int\limits_{B(R)}c_1|\nabla\zeta|^{\alpha}(w+\varepsilon)^{\alpha-1-\nu}\zeta^{s-\alpha}dx. \tag{36}$$

Here we use the symbols c_i, $i = 1, 2, \ldots$, to denote constants depending possibly on n, q, s, α, ν, $\mathcal{K}$ but not on R and ε. Now from (32) and (36) we obtain

$$\int\limits_{B(R)} c_1 |\nabla\zeta|^\alpha (w+\varepsilon)^{\alpha-1-\nu} \zeta^{s-\alpha} dx$$
$$\geq \frac{\nu}{2} \int\limits_{B(R)} \sum_{i=1}^{n} w_{x_i} (A_i(x, \nabla u) - A_i(x, \nabla v))(w+\varepsilon)^{-\nu-1} \zeta^s dx$$
$$+ \int\limits_{B(R)} (|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-\nu} \zeta^s dx. \tag{37}$$

Since the function $w(x)$ is non-negative and $\nu > \alpha - 1$, (37) implies

$$\begin{aligned} c_2 \varepsilon^{\alpha-1-\nu} \int\limits_{B(R)} |\nabla\zeta|^\alpha dx \ \geq \ & \int\limits_{B(R)} \sum_{i=1}^{n} w_{x_i} (A_i(\nabla u) - A_i(\nabla v))(w+\varepsilon)^{-\nu-1} \zeta^s dx \\ & + \int\limits_{B(R)} (|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-\nu} \zeta^s dx. \end{aligned} \tag{38}$$

Now (38) and (31) yield

$$\begin{aligned} c_3 \varepsilon^{\alpha-1-\nu} R^{n-\alpha} \ \geq \ & \int\limits_{B(R)} \sum_{i=1}^{n} w_{x_i} (A_i(\nabla u) - A_i(\nabla v))(w+\varepsilon)^{-\nu-1} dx \\ & + \int\limits_{B(R/2)} (|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-\nu} dx. \end{aligned} \tag{39}$$

Moreover, since $\alpha \geq n$, $u \geq v$ and since $t \mapsto |t|^{q-1}t$ is monotone, the integrals on the right-hand side of (39) are nonnegative and bounded for all $R > 0$. Sending R to ∞ in (39) leads to

$$\int\limits_{B(R_k)\setminus B(R_k/2)} \sum_{i=1}^{n} w_{x_i} (A_i(x, \nabla u) - A_i(x, \nabla v))(w+\varepsilon)^{-\nu-1} \to 0 \tag{40}$$

as well as (if $\alpha > n$)

$$\int\limits_{B(R_k/2)} (|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-\nu} dx \to 0 \tag{41}$$

for all sequences $R_k \to \infty$. But that in turn implies that $u(x) = v(x)$ a.e. on $\mathbb{R}^n$. To cover also the case $n = \alpha$, since I_1 is non-positive, it follows from (32) and (34) that

$$\int\limits_{B(R)} s\mathcal{K}^{1/\alpha} \left(\sum_{i=1}^{n} w_{x_i}(A_i(x, \nabla u) - A_i(x, \nabla v)) \right)^{(\alpha-1)/\alpha} |\nabla \zeta| (w+\varepsilon)^{-\nu} \zeta^{s-1} dx$$
$$\geq \int\limits_{B(R)} (|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-\nu} \zeta^s dx. \tag{42}$$

Estimating the integrand on the left-hand side of (42) by Hölder's inequality, we obtain

$$c_4 \left(\int\limits_{B(R)\setminus B(R/2)} \sum_{i=1}^{n} w_{x_i}(A_i(x, \nabla u) - A_i(x, \nabla v))(w+\varepsilon)^{-\nu-1} dx \right)^{\frac{\alpha-1}{\alpha}}$$
$$\times \left(\int\limits_{B(R)} |\nabla \zeta|^{\alpha} (w+\varepsilon)^{\alpha-1-\nu} dx \right)^{\frac{1}{\alpha}} \geq \int\limits_{B(R/2)} (|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-\nu} dx. \tag{43}$$

We recall the choice of ζ above. Since $w(x)$ is non-negative and $\nu > \alpha - 1$ (43) and (31) yield

$$c_5 \left(\int\limits_{B(R)\setminus B(R/2)} \sum_{i=1}^{n} w_{x_i}(A_i(x, \nabla u) - A_i(x, \nabla v))(w+\varepsilon)^{-\nu-1} dx \right)^{\frac{\alpha-1}{\alpha}}$$
$$\times \varepsilon^{\frac{\alpha-1-\nu}{\alpha}} R^{\frac{n-\alpha}{\alpha}} \geq \int\limits_{B(R/2)} (|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-\nu} dx. \tag{44}$$

It then follows directly from (40) and (44) that

$$\int\limits_{B(R_k/2)} (|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-\nu} dx \to 0 \tag{45}$$

as $R_k \to \infty$, and that in turn yields that $u(x) = v(x)$ on $\mathbb{R}^n$. $\Box$

Proof of Theorem 2. We start by observing that (see, e.g, [9])

$$(|u|^{q-1}u - |v|^{q-1}v)(u-v) \geq c_1|u-v|^{q+1} \tag{46}$$

for any real numbers $q \geq 1$, u and v and $c_1 = 2^{1-q}$. Thus (15) implies

$$\int\limits_{\mathbb{R}^n} \sum_{i=1}^{n} \varphi_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v))dx \geq c_1 \int\limits_{\mathbb{R}^n} (u-v)^q \varphi dx \tag{47}$$

for every non-negative test function $\varphi \in C_0^\infty(\mathbb{R}^n)$. Choose ζ and $\varphi(x) = (w(x)+\varepsilon)^{-\nu}\zeta^s(x)$ as in the proof of Theorem 1, where $w(x) = u(x) - v(x)$. The positive constants $s \geq \alpha$ and $\alpha - 1 > \nu > 0$ will be chosen below. Integrating by parts in (47) gives

$$\begin{aligned} -\nu \int\limits_{B(R)} \sum_{i=1}^{n} w_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v))(w+\varepsilon)^{-\nu-1}\zeta^s dx \\ +s \int\limits_{B(R)} \sum_{i=1}^{n} \zeta_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v))(w+\varepsilon)^{-\nu}\zeta^{s-1} dx \\ \equiv I_1 + I_2 \geq c_1 \int\limits_{B(R)} w^q(w+\varepsilon)^{-\nu}\zeta^s dx. \end{aligned} \tag{48}$$

As in the previous proof we estimate I_2 in terms of I_1 and arrive at

$$\begin{aligned} |I_2| \leq \frac{\nu}{2} \int\limits_{B(R)} \sum_{i=1}^{n} w_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v))(w+\varepsilon)^{-\nu-1}\zeta^s dx \\ + \int\limits_{B(R)} c_2|\nabla\zeta|^\alpha (w+\varepsilon)^{\alpha-1-\nu}\zeta^{s-\alpha} dx. \end{aligned} \tag{49}$$

Moreover, inequality

$$\begin{aligned} \int\limits_{B(R)} c_2|\nabla\zeta|^\alpha (w+\varepsilon)^{\alpha-1-\nu}\zeta^{s-\alpha} dx \geq c_1 \int\limits_{B(R)} w^q(w+\varepsilon)^{-\nu}\zeta^s dx \\ +\frac{\nu}{2} \int\limits_{B(R)} \sum_{i=1}^{n} w_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v))(w+\varepsilon)^{-\nu-1}\zeta^s dx \end{aligned} \tag{50}$$

immediately follows from (48) and (49). Next we estimate the integrand on the left-hand side of (50) by Young's inequality (35) with $\rho = \frac{c_1}{2}$, $\beta = \frac{q-\nu}{q-\alpha+1}$,

$$\mathcal{A} = (w+\varepsilon)^{\alpha-\nu-1}\zeta^{\frac{s(\alpha-1-\nu)}{q-\nu}}, \tag{51}$$

and

$$\mathcal{B} = c_2|\nabla\zeta|^{\alpha}\zeta^{\frac{s(q-\alpha+1)}{q-\nu}-\alpha}, \tag{52}$$

and obtain

$$\begin{aligned} \frac{c_1}{2}\int\limits_{B(R)}(w+\varepsilon)^{q-\nu}\zeta^s dx + c_3\int\limits_{B(R)}|\nabla\zeta|^{\frac{\alpha(q-\nu)}{q-\alpha+1}}\zeta^{s-\frac{\alpha(q-\nu)}{q-\alpha+1}}dx \\ \geq c_1\int\limits_{B(R)} w^q(w+\varepsilon)^{-\nu}\zeta^s dx \\ +\frac{\nu}{2}\int\limits_{B(R)}\sum_{i=1}^{n} w_{x_i}(A_i(x,\nabla u)-A_i(x,\nabla v))(w+\varepsilon)^{-\nu-1}\zeta^s dx. \end{aligned} \tag{53}$$

Since the last term on the right-hand side of (53) is non-negative, this yields

$$\begin{aligned} \frac{c_1}{2}\int\limits_{B(R)}(w+\varepsilon)^{q-\nu}\zeta^s dx + c_3\int\limits_{B(R)}|\nabla\zeta|^{\frac{\alpha(q-\nu)}{q-\alpha+1}}\zeta^{s-\frac{\alpha(q-\nu)}{q-\alpha+1}}dx \\ \geq c_1\int\limits_{B(R)} w^q(w+\varepsilon)^{-\nu}\zeta^s dx. \end{aligned} \tag{54}$$

In (54) we can send $\varepsilon \to 0$. This is justified by Lebesgue's theorem (see, e.g. p. 303 in [9]), and for $s \geq \alpha\frac{q-\nu}{q-\alpha+1}$ it leads to

$$c_4\int\limits_{B(R)}|\nabla\zeta|^{\frac{\alpha(q-\nu)}{q-\alpha+1}}dx \geq \int\limits_{B(R)} w^{q-\nu}\zeta^s dx. \tag{55}$$

We chose $\zeta(x)$ as in the proof of Theorem 1. Then (55) and (31) yield

$$c_5 R^{n-\frac{\alpha(q-\nu)}{q-\alpha+1}} \geq \int\limits_{B(R)} w^{q-\nu}dx. \tag{56}$$

It is easy to see that for $1 < q < \frac{n(\alpha-1)}{n-\alpha}$ and sufficiently small ν, the inequality

$$n - \frac{\alpha(q-\nu)}{q-\alpha+1} < 0 \tag{57}$$

holds. Sending $R \to +\infty$ on the left-hand side of (56) and observing (57) gives

$$\int\limits_{\mathbb{R}^n} w^{q-\nu} dx = 0, \tag{58}$$

where $q > \nu$. But then $w(x) = 0$ a.e. on $\mathbb{R}^n$. Thus, for $1 < q < \frac{n(\alpha-1)}{n-\alpha}$, we have proved that $u(x) = v(x)$ a.e. on $\mathbb{R}^n$. This proves Theorem 2 in the case $q < \frac{n(\alpha-1)}{n-\alpha}$.

To treat the case $q = \frac{n(\alpha-1)}{n-\alpha}$, we borrow an idea from [19], which boils down to estimating the integral

$$\int_{B(R)} w^q \zeta^s dx \tag{59}$$

by expoiting (53) from above in the way suggested in [12]. To this end, we set $\varphi(x) = \zeta^s(x)$ in (15). Then (46) implies

$$s \int\limits_{B(R)} \sum_{i=1}^{n} \zeta_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v))\zeta^{s-1} dx \geq c_1 \int\limits_{B(R)} w^q \zeta^s dx. \tag{60}$$

Since

$$\sum_{i=1}^{n} \zeta_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v)) \leq |\nabla\zeta| \left(\sum_{i=1}^{n} (A_i(x,\nabla u) - A_i(x,\nabla v))^2 \right)^{\frac{1}{2}}, \tag{61}$$

using the condition (10) on the coefficients of the operator $A(w)$, (60) and (61) lead to

$$c_1 \int\limits_{B(R)} w^q \zeta^s dx$$

$$\leq s\mathcal{K}^{1/\alpha} \int\limits_{B(R)} \left(\sum_{i=1}^{n} w_{x_i}(A_i(x,\nabla u) - (A_i(x,\nabla v)) \right)^{\frac{\alpha-1}{\alpha}} |\nabla\zeta|\zeta^{s-1} dx. \tag{62}$$

Estimating the right-hand side of (62) by Hölder's inequality, it is easy to see that the inequality

$$c_1 \int\limits_{B(R)} w^q \zeta^s dx \le s\mathcal{K}^{1/\alpha} \left(\int\limits_{B(R)} |\nabla \zeta|^\alpha (w+\varepsilon)^{(\alpha-1)(\nu+1)} \zeta^{s-\alpha} dx \right)^{1/\alpha}$$

$$\times \left(\int\limits_{B(R)} \sum_{i=1}^n w_{x_i} (A_i(x,\nabla u) - (A_i(x,\nabla v))(w+\varepsilon)^{-\nu-1} \zeta^s dx \right)^{\frac{\alpha-1}{\alpha}} \qquad (63)$$

holds for any $\varepsilon > 0$. Further, since the inequality

$$\int\limits_{B(R)} |\nabla \zeta|^\alpha (w+\varepsilon)^{(\alpha-1)(\nu+1)} \zeta^{s-\alpha} dx$$

$$\le \left(\int\limits_{B(R)\setminus B(R/2)} (w+\varepsilon)^{d(\alpha-1)(1+\nu)} \zeta^s dx \right)^{\frac{1}{d}} \left(\int\limits_{B(R)} |\nabla \zeta|^{\frac{\alpha d}{d-1}} \zeta^{s-\frac{\alpha d}{d-1}} dx \right)^{\frac{d-1}{d}} \qquad (64)$$

is valid for any $d > 1$, by choosing any sufficiently small ν from the interval $(0, \alpha-1) \cap (0, \frac{q-\alpha+1}{\alpha-1})$, and the parameter $d = \frac{q}{(\alpha-1)(1+\nu)}$ such that $d(\alpha-1)(1+\nu) = q$, (63) and (64) yield

$$c_1 \int\limits_{B(R)} w^q \zeta^s dx$$

$$\le s\mathcal{K}^{1/\alpha} \left(\int\limits_{B(R)} |\nabla \zeta|^{\frac{\alpha d}{d-1}} \zeta^{s-\frac{\alpha d}{d-1}} dx \right)^{\frac{d-1}{\alpha d}} \left(\int\limits_{B(R)\setminus B(R/2)} (w+\varepsilon)^q \zeta^s dx \right)^{\frac{1}{\alpha d}}$$

$$\times \left(\int\limits_{B(R)} \sum_{i=1}^n w_{x_i} (A_i(x,\nabla u) - (A_i(x,\nabla v))(w+\varepsilon)^{-\nu-1} \zeta^s dx \right)^{\frac{\alpha-1}{\alpha}}. \qquad (65)$$

Estimating now the last term on the right-hand side of (65) by (53), we obtain

$$c_1 \int\limits_{B(R)} w^q \zeta^s dx \le s\mathcal{K}^{\frac{1}{\alpha}} \left(\int\limits_{B(R)} |\nabla\zeta|^{\frac{\alpha d}{d-1}} \zeta^{s-\frac{\alpha d}{d-1}} dx \right)^{\frac{d-1}{\alpha d}}$$

$$\times \left(\int\limits_{B(R)\setminus B(R/2)} (w+\varepsilon)^q \zeta^s dx \right)^{\frac{1}{\alpha d}} \left(c_4 \int\limits_{B(R)} |\nabla\zeta|^{\alpha\frac{q-\nu}{q-\alpha+1}} \zeta^{s-\alpha\frac{q-\nu}{q-\alpha+1}} dx \right.$$

$$\left. + \frac{c_1}{\nu} \int\limits_{B(R)\setminus B(R/2)} (w+\varepsilon)^{q-\nu} \zeta^s dx - \frac{2c_1}{\nu} \int\limits_{B(R)} w^q (w+\varepsilon)^{-\nu} \zeta^s dx \right)^{\frac{\alpha-1}{\alpha}}. \qquad (66)$$

In (66), passing to the limit as $\varepsilon \to 0$ as justified by Lebesgue's theorem (see, e.g., [9], p. 303), gives

$$\int\limits_{B(R)} w^q \zeta^s dx \le c_6 \left(\int\limits_{B(R)\setminus B(R/2)} w^q \zeta^s dx \right)^{\frac{1}{\alpha d}}$$

$$\times \left(\int\limits_{B(R)} |\nabla\zeta|^{\frac{\alpha d}{d-1}} \zeta^{s-\frac{\alpha d}{d-1}} dx \right)^{\frac{d-1}{\alpha d}} \left(\int\limits_{B(R)} |\nabla\zeta|^{\alpha\frac{q-\nu}{q-\alpha+1}} \zeta^{s-\alpha\frac{q-\nu}{q-\alpha+1}} dx \right)^{\frac{\alpha-1}{\alpha}}. \qquad (67)$$

Therefore, for sufficiently large s, (67) implies

$$\left(\int\limits_{B(R/2)} w^q dx \right)^{\frac{\alpha d-1}{\alpha d}} \le c_7 \left(\int\limits_{B(R)} |\nabla\zeta|^{\frac{\alpha d}{d-1}} dx \right)^{\frac{d-1}{\alpha d}} \left(\int\limits_{B(R)} |\nabla\zeta|^{\alpha\frac{q-\nu}{q-\alpha+1}} dx \right)^{\frac{\alpha-1}{\alpha}}. \qquad (68)$$

Now choose in (68) the function $\zeta(x)$ as the usual cut-off function. Then (31) and (68) yield

$$\left(\int\limits_{B(R/2)} w^q dx \right)^{\frac{\alpha d-1}{\alpha d}} \le c_8 R^p, \qquad (69)$$

where

$$p = \frac{n - p_1}{p_1} + \frac{(\alpha - 1)(n - p_2)}{\alpha}, \quad p_1 = \frac{\alpha d}{d - 1}, \qquad p_2 = \frac{\alpha(q - \nu)}{q - \alpha + 1}. \tag{70}$$

It is easy to calculate that for $1 < \alpha < n$

$$p = \frac{(n - \alpha)(\alpha q - \alpha + 1 - \nu(\alpha - 1))}{\alpha q(q - \alpha + 1)} \left(q - \frac{n(\alpha - 1)}{n - \alpha} \right). \tag{71}$$

Further, for $n > \alpha$ and $q = \frac{n(\alpha-1)}{n-\alpha}$, it follows from (69) and (71) that the integral

$$\int_{\mathbb{R}^n} w^q dx \tag{72}$$

is bounded. Therefore, due to monotonicity, the relation

$$\int\limits_{B(R_k)\setminus B(R_k/2)} w^q dx \to 0 \tag{73}$$

holds for any sequence $R_k \to \infty$. On the other hand, for sufficiently large s it follows from (67) that

$$\int\limits_{B(R/2)} w^q dx \le c_6 \left(\int\limits_{B(R)\setminus B(R/2)} w^q dx \right)^{\frac{1}{\alpha d}}$$

$$\times \left(\int\limits_{B(R)} |\nabla \zeta|^{\frac{\alpha d}{d-1}} dx \right)^{\frac{d-1}{\alpha d}} \left(\int\limits_{B(R)} |\nabla \zeta|^{\alpha \frac{q-\nu}{q-\alpha+1}} dx \right)^{\frac{\alpha-1}{\alpha}}. \tag{74}$$

In (74), choosing the function $\zeta(x)$ of the form indicated above and observing (31) gives

$$\int\limits_{B(R/2)} w^q dx \le c_9 R^p \left(\int\limits_{B(R)\setminus B(R/2)} w^q dx \right)^{\frac{1}{\alpha d}}. \tag{75}$$

Finally, for $n > \alpha$, $q = \frac{n(\alpha-1)}{n-\alpha}$ and any sufficiently small $\nu \in (0, \alpha - 1) \cap (0, \frac{q-\alpha+1}{\alpha-1})$, we have from (71),(73) and (75) the relation

$$\int\limits_{B(R_k/2)} w^q dx \to 0 \tag{76}$$

for any sequence $R_k \to \infty$. Now this implies that

$$\int_{\mathbb{R}^n} w^q dx = 0 \tag{77}$$

and hence that $u(x) = v(x)$ on $\mathbb{R}^n$.$\square$

Proof of Theorem 3. The proof is indirect, by contradiction. Assume that there exists an entire weak solution (u, v) of inequality (1) on $\mathbb{R}^n$ such that $u(x) \geq v(x)$ satisfying (18). Then by (46) we have from (15) the inequality

$$\int_{\mathbb{R}^n} \sum_{i=1}^{n} \varphi_{x_i}(A_i(x, \nabla u) - A_i(x, \nabla v))dx \geq c_1 \int_{\mathbb{R}^n} (u - v)^q \varphi dx \tag{78}$$

for every non-negative smooth test function $\varphi \in C_0^\infty(\mathbb{R}^n)$. Without loss of generality, we substitute $\varphi(x) = (w(x) + \varepsilon)^{-\nu}\zeta^s(x)$ as a test function in (78), where $w(x) = u(x) - v(x)$, ζ is the standard cut-off function from the proof of Theorem 1, and the positive constants $s \geq \alpha$ and $q > \nu > 0$ will be chosen below. Integrating by parts in (78) and repeating the arguments in the proof of Theorem 2 we arrive again at (56)

$$c_5 R^{n - \frac{\alpha(q-\nu)}{q-\alpha+1}} \geq \int_{B(R)} w^{q-\nu} dx,$$

which holds for any $\nu \in (0, \alpha - 1]$ and which contradicts hypothesis (18) of Theorem 3.

Proof of Theorem 4. Also this proof is indirect by reduction to a contradiction. If there exists an entire weak solution (u, v) of inequality (1) on $\mathbb{R}^n$ such that $u(x) \geq v(x)$ and (21) holds, then

$$\int_{\mathbb{R}^n} \sum_{i=1}^{n} (A_i(x, \nabla u) - A_i(x, \nabla v))\varphi_{x_i} dx \geq \int_{\mathbb{R}^n} (|u|^{q-1}u - |v|^{q-1}v)\varphi dx, \tag{79}$$

for every non-negative smooth test function $\varphi \in C_0^\infty(\mathbb{R}^n)$. Similar to the previous proofs, we set $\varphi(x) = (w + \varepsilon)^{1-\alpha}\zeta^s(x)$ as a test function in (79), where $w(x) = u(x) - v(x)$, $s \geq \alpha$ and ζ is the usual cut-off function. Then an inspection of (39) for $\nu = \alpha - 1$ leads to

$$c_5 R^{n-\alpha} \geq \int_{B(R)\cap\{x\in\mathbb{R}^n :\, u(x)\neq v(x)\}} (|u|^{q-1}u - |v|^{q-1}v)(u - v)^{1-\alpha} dx \tag{80}$$

and this contradicts (21).

Proof of Theorem 5. Again we prove this by contradiction. Suppose there exists an entire weak solution of (1). We follow the arguments in the proof of Theorem 4. For $q = 1$ and $\alpha = 2$ relation (80) becomes

$$\limsup_{R\to+\infty} R^{2-n} \int\limits_{\{|x|<R\}\cap\{x:\,u(x)\neq v(x)\}} 1\ dx \leq c_3, \tag{81}$$

but if $u(x) > v(x)$ in $\mathbb{R}^n$, then the left hand side of (81) is infinite, a contradiction.

Proof of Theorem 6. It first follows from (16) that the inequality

$$-\int\limits_{\mathbb{R}^n} \sum_{i=1}^{n} \varphi_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v))dx \geq \int\limits_{\mathbb{R}^n} (|u|^{q-1}u - |v|^{q-1}v)\varphi dx \tag{82}$$

holds for every non-negative smooth function $\varphi \in C^\infty(\mathbb{R}^n)$ with compact support. We choose $\varphi(x) = w^\nu(x)\zeta^s(x)$ as a test function, where $w(x) = \max\{u(x) - v(x), 0\}$, ζ is the standard cut-off function and the positive constants $s \geq \alpha$ and $\nu > 1$ will be chosen below. Then, by (46), we arrive at

$$-\int\limits_{\mathbb{R}^n} \sum_{i=1}^{n} (w^\nu\zeta^s)_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v))dx \geq c_1 \int\limits_{\mathbb{R}^n} w^{q+\nu}\varphi dx. \tag{83}$$

Integrating this by parts gives

$$\begin{aligned}
&-\nu \int\limits_{B(R)} \sum_{i=1}^{n} w_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v))w^{\nu-1}\zeta^s dx \\
&-s \int\limits_{B(R)} \sum_{i=1}^{n} \zeta_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v))w^\nu\zeta^{s-1} dx \\
&\qquad\qquad \equiv I_1 + I_2 \geq c_1 \int\limits_{B(R)} w^{q+\nu}\zeta^s dx.
\end{aligned} \tag{84}$$

As above, we estimate I_2 in terms of I_1. Since

$$|I_2| = \left| s \int\limits_{B(R)} \sum_{i=1}^{n} \zeta_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v)) w^{\nu} \zeta^{s-1} dx \right|$$

$$\leq \int\limits_{B(R)} s \left(\sum_{i=1}^{n} (A_i(x,\nabla u) - (A_i(x,\nabla v))^2 \right)^{\frac{1}{2}} |\nabla \zeta| w^{\nu} \zeta^{s-1} dx, \tag{85}$$

we can proceed as in the proof of Theorem 1 and use condition (10) on the coefficients of the operator A to obtain

$$|I_2| \leq$$

$$\int\limits_{B(R)} s\mathcal{K}^{1/\alpha} |\nabla \zeta| w^{\nu} \zeta^{s-1} \left(\sum_{i=1}^{n} w_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v)) \right)^{\frac{\alpha-1}{\alpha}} dx. \tag{86}$$

For $\alpha > 1$ we estimate the integrand on the right-hand side of (86) by Young's inequality (35) with $\rho = \frac{\nu}{2}$, $\beta = \alpha$,

$$\mathcal{A} = w^{\frac{(\alpha-1)(\nu-1)}{\alpha}} \zeta^{\frac{s(\alpha-1)}{\alpha}} \left(\sum_{i=1}^{n} w_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v)) \right)^{\frac{\alpha-1}{\alpha}} \tag{87}$$

and

$$\mathcal{B} = s\mathcal{K}^{1/\alpha} |\nabla \zeta| \zeta^{\frac{s}{\alpha}-1} w^{\frac{\alpha-1+\nu}{\alpha}}, \tag{88}$$

and we arrive at

$$|I_2| \leq \frac{\nu}{2} \int\limits_{B(R)} \sum_{i=1}^{n} w_{x_i}(A_i(x,\nabla u) - A_i(x,\nabla v)) w^{\nu-1} \zeta^{s} dx$$

$$+ \int\limits_{B(R)} c_2 |\nabla \zeta|^{\alpha} w^{\alpha-1+\nu} \zeta^{s-\alpha} dx. \tag{89}$$

Since $I_1 \leq 0$, (89) and (84) imply

$$c_3 \int\limits_{B(R)} |\nabla \zeta|^{\alpha} w^{\alpha-1+\nu} \zeta^{s-\alpha} dx \geq \int\limits_{B(R)} w^{q+\nu} \zeta^{s} dx \tag{90}$$

for $\alpha > 1$ and from (10) and (84) for $\alpha = 1$.

Further, since $q > \alpha - 1$, choosing $s = \frac{\alpha(q+\nu)}{q-\alpha+1}$ such that $(s-\alpha)\frac{q+\nu}{\alpha-1+\nu} = s$ and estimating the left-hand side of (90) by Hölder's inequality we obtain

$$c_4 \left(\int\limits_{B(R)} |\nabla\zeta|^{\frac{\alpha(q+\nu)}{q-\alpha+1}} \right)^{\frac{q-\alpha+1}{q+\nu}} \left(\int\limits_{B(R)} w^{q+\nu}\zeta^s dx \right)^{\frac{\alpha-1+\nu}{q+\nu}} \geq \int\limits_{B(R)} w^{q+\nu}\zeta^s dx. \tag{91}$$

Recalling (31) we arrive at

$$c_6 R^{n-\frac{\alpha(q+\nu)}{q-\alpha+1}} \geq \int\limits_{B(R)} w^{q+\nu} dx \tag{92}$$

which holds for any $R > 0$. Now we choose in (92) the parameter ν sufficiently large such that

$$n - \frac{\alpha(q+\nu)}{q-\alpha+1} < 0 \tag{93}$$

holds. The sending $R \to \infty$ in (92) implies

$$\int_{\mathbb{R}^n} w^{q+\nu} dx = 0, \tag{94}$$

that is $w(x) = 0$ or $u(x) \leq v(x)$ a.e. on $\mathbb{R}^n$. $\square$

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