Some nonconvex shape optimization problems

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I. Minimizing paths, the opaque square

The following problem was pointed out to me by Fred Almgren. Imagine a piece of land, which is quadratic and of size one. The owner of the land is an unpleasant person. He does not want his neighbours to be able to communicate with each other by talking across his property. One way to reach his goal is to build a fence around the land which would block any line of sight across it. Moreover, the landowner is somewhat stingy and does not want to spend more money than necessary on building the fence. In fact a fence on three sides of his property would serve the same purpose and be cheaper. In that case a neighbour on the free side could look inside his property but not across it to a neighbour on any of the other three sides. What is the shortest fence that would have the desired function of blocking every line of sight across the property?

Although this shape optimization problem looks fairly simple it offers a couple of surprises even for mathematicians working on shape problems. Clearly we want the vision across each of the four corners to be blocked, so an optimal fence will contain all four corners of the square. So building two diagonal fences of total length 2 times $\sqrt{2}$, i.e. ≈ 2.82, would be preferable to building on three sides around. (Never mind the discomfort that this fence across his land may cause to the owner, he is just happy to get away cheaper.)

Finding the shortest path that connects four points is a well-understood problem in optimization. Its solution is the Steiner tree depicted in Figure 1.1, and its length is $1+\sqrt{3}$, that is ≈ 2.73. Now this tree has only two reflection symmetries, although the square has four. The Steiner tree is not symmetric across the diagonals. There is, however, an even shorter fence, the first record of which I have found in [34]. According to [32] the same solution was found by Maurice Poirier, a secondary school teacher in Orléans, Ontario, who might have been inspired by a test for mathematically gifted school children. The test can be found in [60] and is quoted in [32]. The shortest known fence is depicted in Figure 1.2 and cosists of two components, namely a Steiner tree connecting three corners. This tree blocks vision from its convex hull, so the remainder of the fence is the shortest line from the fourth corner to the triangle generated by the Steiner tree. It has length $(2+\sqrt{3})/\sqrt{2} \approx 2.64$, and it features only one reflection symmetry.

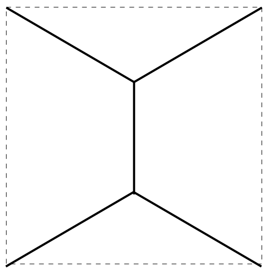

Fig.1.1. The Steiner tree

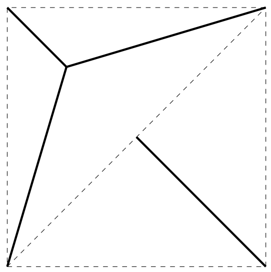

Fig.1.2. The shortest known fence

In [39] I outlined a proof that the fence depicted in Figure 1.2 is a curve which minimizes one-dimensional Hausdorff measure in the class of all doubly connected curves C which have the property that every straight line intersecting the square D also intersects C. To fix some ideas let me give two definitions.

Definition 1.1:
Let $C \subset \mathbb{R}^n$. Then the 1-dimensional Hausdorff measure of C is defined as

$$\mathcal{H}^1(C) := \lim_{\delta \to 0} \inf\{\sum_{i=1}^{\infty} \mathrm{diam}(U_i)^1 \mid U_i \text{ cover } C \text{ and have diam} \leq \delta \ \}$$

Definition 1.2:
Let E and F be subsets of $\mathbb{R}^n$. Then their Euclidean δ-neighbourhood is denoted by $[E]_\delta$ and $[F]_\delta$, and their Hausdorff distance is defined as

$$\delta(E,F) := \inf\{\delta \mid E \subset [F]_\delta \text{ and } F \subset [E]_\delta \ \}.$$

This distance gives rise to a metric, the Hausdorff metric.

Theorem 1.1 [22]:
For each maximum number $k \in \mathbb{N}$ of components and for every convex compact $D \subset \mathbb{R}^n$, there exists a curve $C_k \subset D$ minimizing one-dimensional Hausdorff-measure under the geometric side constraint, that every straight line intersecting D also intersects the curve.

This was apparently first shown in [22] but may be more accessible in [24, p.39]. Let me sketch the proof. Clearly the Hausdorff-measure $\mathcal{H}^1(C)$ is bounded below and thus there exists a minimizing sequence C^j of admissible sets, each consisting of k pairwise disjoint connected components $C_1^j, \ldots, C_k^j$. Due to Blaschke's selection theorem [24, p.37], for $m = 1, \ldots, k$ and after passing to a subsequence the sequence of connected sets C_m^j converges in Hausdorff metric to a connected limit set C_m^∞. Fortunately Hausdorff measure is lower semicontinuous along this subsequence, that is

$$\mathcal{H}^1(C_m^\infty) \leq \mathcal{H}^1(C_m^j). \tag{1.1}$$

The union over $m = 1, \ldots, k$ of these C_m^∞ is a minimizer C_k, since by the subadditivity of $\mathcal{H}^1$ we have

$$\mathcal{H}^1(C_k) \leq \sum_{m=1}^{k} \mathcal{H}^1(C_m^j) = \mathcal{H}^1(C^j). \tag{1.2}$$

Note first that the last equality follows for each fixed j from the pairwise disjointness of the sets C_m^j, second that the geometric constraint is preserved under convergence of sets in the Hausdorff metric, so that C_k blocks vision just like each of the C^j did, and third that it is conceivable that C_k has less than k components, because some of the C_m^∞ might have nonempty pairwise intersection. This proves the existence result of a minimizing C_k for every k. To verify the geometric constraint one simply has to realize that the points x_j of intersection of a given line of sight with C^j converge to a point both in C^∞ and on the given line.

Now C_1 and C_2 are shown in Figures 1.1 and 1.2, and it is conjectured that $C_k = C_2$ for any natural number $k \geq 2$, and not only for even prime numbers.

What happens as $k \to \infty$? I do not know of any existence result in the class of admissible curves with a countable number of components. The difficulty of proving this lies in the fact that the semicontinuity result (1.2) is no longer true if the sets C^j have a countable number of components. A counterexample which I learned from G.Dal Maso is provided by the following sequence of subsets C^j of the unit interval $[0,1]$ in $\mathbb{R}$.

$$\begin{aligned} C^1 &= \left[0, \frac{1}{2}\right], \\ C^2 &= \left[0, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right], \end{aligned}$$

$$C^3 = \left[0, \frac{1}{8}\right] \cup \left[\frac{1}{4}, \frac{3}{8}\right] \cup \left[\frac{1}{2}, \frac{5}{8}\right] \cup \left[\frac{3}{4}, \frac{7}{8}\right] \quad \text{etc.}$$

Clearly the Hausdorff distance of C^j to $[0,1]$ is 2^{-j} so that $C^j \to C_\infty = [0,1]$ in Hausdorff distance, but $\mathcal{H}^1(C_\infty) = 1 > 0.5 = \mathcal{H}^1(C^j)$. We shall see a similar example below.

Why is C_2 from Figure 1.2 optimal in the class of doubly connected curves? The optimal C must contain all four corners, otherwise one could look across a corner. Therefore its two connected but disjoint components C^1 and C^2 must emanate from the corners of the square. Several cases are conceivable. If C^1 contains exactly two diagonal corners, then C^2 must cross C^1, a contradiction to being disjoint. If C^1 contains exactly two adjacent corners, then the convex hulls of C^1 and C^2 must have nonempty intersection. Let z be a point from this intersection. Connecting all four corners with straight line segments to z will yield a shorter, and simply connected curve; again a contradiction. Thus the case that C^1 (or equivalently C^2) contains exactly three corners remains to be treated. If C^1 remains in the triangle spanned by its three corners, it must be the Steiner tree depicted in Figure 1.2, and then C^2 is half a diagonal, so that they make up the two components in Figure 1.2. Finally the case that C^1 exceeds the triangle spanned by its three corners a, b and c was neglected in [39] and can be treated as follows. I am indebted to Marino Belloni for pointing this out to me. To be disconnected from C^1 but still block vision, C^2 must touch the convex hull of C^1 but not C^1 in some point g. So g lies on a line segment, the endpoints of which, e and f, are elements of C^1, see Figure 1.3.

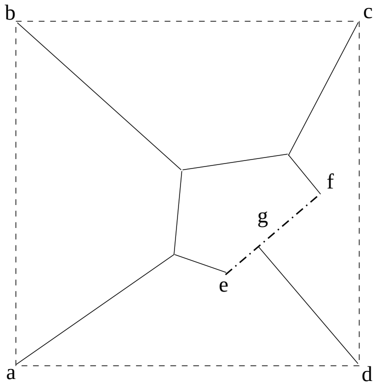

Fig.1.3. Not an optimal situation

Since C^2 minimizes distance from the fourth corner d to the convex hull of C^1, the segments ef and dg must be orthogonal to each other. Now I move e and f away from the fourth corner d in the direction of dg. This will shorten the path connecting a, b, c, e and f more than it will lengthen the one from d to g. This completes the proof that the solution depicted in Figure 1.2 is optimal in the class of curves with at most two components.

There are several ways to generalize this problem. One can look at the three-dimensional cube instead of a twodimensional square and try to build a twodimensional (minimal) surface which blocks vision through the cube. A rather small surface consists of a rounded square in the middle with surfaces extending from the edges of the small rounded square to the large original cube. It shows up, when you dip a cubic wire frame into soap. An even better surface was calculated by Brakke, see [9], and yields an interesting surface which forms a tunnel within it. Of course to pass through the tunnel, you cannot go straight.

Another way is to pass from a square to other polygons or even to a circle in two dimensions. The circle has an interesting "application". In [61] Sherlock Holmes has to find an underground tunnel very fast. It is a straight tunnel that runs under a lawn within a given radius 1 of a given point, and the only way to find it is by digging. More realistic seems to be the application of locating a straight telephone line in the ground, provided it is within radius 1 of a given marker [23]. In fact, when I presented this problem in Oberwolfach in 1995, M.Plum asked me what could be said about the circle. From the experience with the square we expect the optimal fence, trench or simply curve to contain all the boundary points of the circle. Since those form are a set of length 2π, the optimal curve should have at least (one-dimensional Hausdorff) measure 2π? The answer is again not intuitive even for the convex analyst.

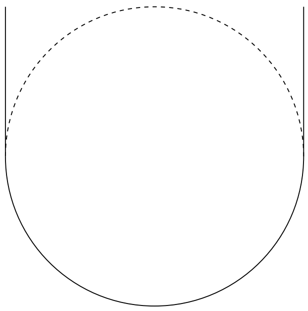

Fig.1.4. How to find a telephone line

Figure 1.4 shows a solution that, according to [23], has been known to M.Magidor in 1974. It leaves the unit disk, has total length $2 + \pi =\approx 5.14$, and is the shortest simply connected curve which intersects all the lines that intersect the circle [22]. If one admits multiply connected curves which may leave the disk, one can do even better and come up with the nowadays apparently best known doubly connected one of length ≈ 4.8189 [22 p.264, St. p.113], a triply connected one of length $1 + \sqrt{3} + 2\pi/3 \approx 4.7998$ and so on.

When I saw those solutions, I wondered if one could reach similar results under the additional constraint that the (possibly multiply connected) admissible curves C had to stay inside the unit disk. One way to approach this problem consists of approximating the disk by a regular polygon with many corners. Clearly, for a general polygon, with many corners we cannot expect to prove that a certain curve is optimal. But we can make good and maybe numerically supported guesses. Incidentally, computer scientists are studying the complexity of such a problem, but they call the admissible curves "opaque forests"of a polygon, see [1,19,57]. Maybe the term "forest" comes from the fact that it tends to be made up of Steiner "trees". To investigate a regular polygon with many corners, it is instructive to look first at polygons with few corners and to draw conclusions from them. Clearly, an equilateral triangle has the Steiner tree in Figure 1.5 as an optimal solution.

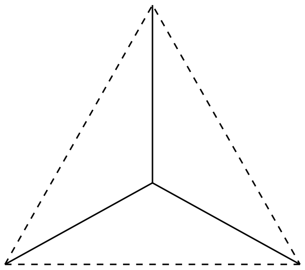

Fig.1.5. Steiner tree of length $\sqrt{3} \approx 1.732$

Incidentally, do you know how to construct the so-called Torricelli point, the meeting point of the three segments, in a general triangle and how to prove that the lines meet in general under the angle $2\pi/3$? This is an old (1640) elementary geometric problem, the solution of which can be traced back to Torricelli, a student of Galileo Galilei and to Cavalieri [14, p.2], but its solution is not so easy to find in textbooks on the calculus of variations. A reference is [14, p.103, 16, Ch. VII] or the one written up for high school students 33].

The best known curve for a regular square was given above in Figure 1.2, the one for a pentagon looks like the one on the left in Figure 1.6, while the one on the right had been suggested by one of the participants of this summer school.

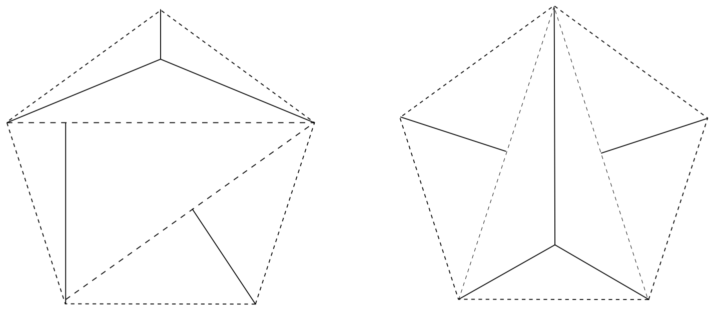

Fig.1.6. Good curves of length ≈ 3.528 [23] and ≈ 3.580 [51]

When we reach the hexagon, the Toricelli point creeps into a corner and the best known even at most doubly connected curve becomes simply connected and connects four boundary points, see Figure 1.7 left. Again the one on the right was brought up during the summer school but is not optimal.

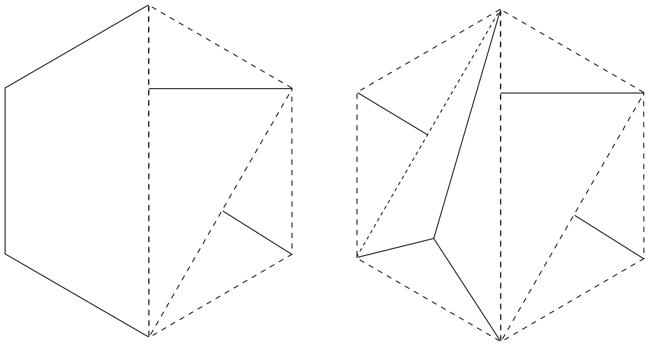

Fig.1.7. Good curves of length $(7+\sqrt{3})/2 \approx 4.366$ [23] and ≈ 4.512 [51]

Now we can extrapolate to a regular $2N$-gon, whose corners lie on the unit circle, and do the following construction: We set $C_0 =$ half of the boundary of the polygon, with P_0 and P_N as endpoints, and suppose that the remaining $(N-1)$ corners are numbered $P_1, P_2, \ldots, P_{N-1}$ in consecutive order. For $i = 1, \ldots, N-1$ let C_i be the straight line segment that connects P_i to the closest point Q_i on the convex hull of $\left(\bigcup_{j=1}^{i} C_j\right)$, see Figure 1.8.

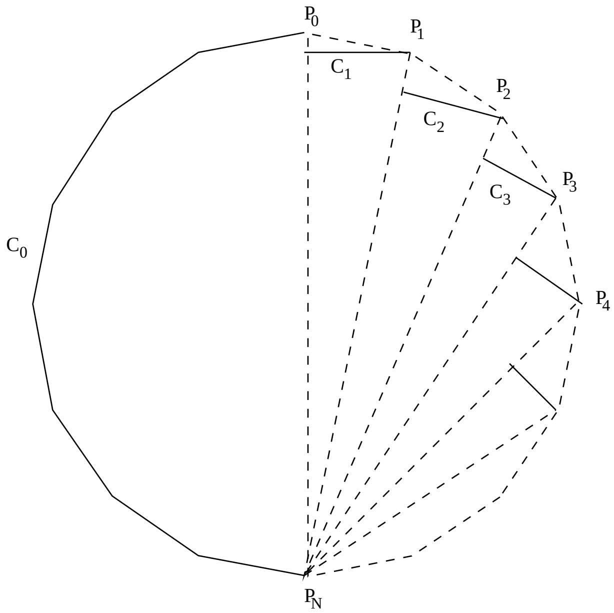

Fig.1.8. Best known solution for a $2N$–gon

Finally set $C_N := \left(\bigcup_{j=0}^{N-1} C_j\right)$. C_N intersects every straight line that crosses the polygon and a fortiori the disk of radius $1-\sin(\pi/2N)$, because the latter is contained in the polygon. Moreover C_N stays inside the closed unit disk. As the number of corners N tends to ∞, we may expect a limit set C_∞ with a countable number of components, which intersects every straight line that intersects the unit disk. So let us calculate the length ℓ_N of C_N by doing some trigonometry.

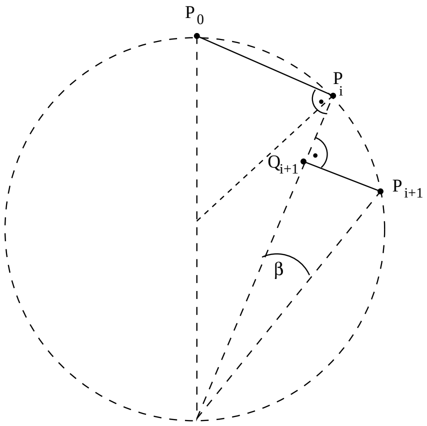

Fig.1.9. Computation of ℓ_N

Let us denote the left endpoint of C_{i+1} by Q_{i+1}. The angle β between P_iP_N and $P_{i+1}P_N$ is $\beta = \pi/2N$, since the angle between P_0P_N and P_iP_N is $i\pi/2N$. The length of the segment $\overline{P_iP_N}$ can be calculated using Thales' Theorem as $\overline{P_iP_N} = 2\cos(i\pi/2N)$, so $\sin\beta = \overline{Q_{i+1}P_{i+1}} \;/\; \overline{P_{i+1}P_N}$ or length of $C_{i+1} = \sin(\pi/2N)\cdot 2\cos((i+1)\pi/2N)$ for $i = 1, \ldots, N-1$.
To get the length of all the arcs we have to add them up

$$\ell_N = \text{ length of } C_0 + \sum_{i=1}^{N-1} \text{ length of } C_i \quad .$$

Now the length of C_0 is $N\cdot\overline{P_0P_1} = N\cdot 2\sin(\pi/2N)$, and so

$$\ell_N = 2N\ \sin\left(\frac{\pi}{2N}\right) + \sum_{j=1}^{N-1} \sin\left(\frac{\pi}{2N}\right)\cdot 2\cos\left(\frac{j\pi}{2N}\right)$$

or, observing that $\cos\pi/2 = 0$,

$$l_N = 2N\ \sin\left(\frac{\pi}{2N}\right)\left(1 + \frac{1}{N}\sum_{j=1}^{N}\cos\left(\frac{j\pi}{2N}\right)\right) \quad .$$

But $\frac{1}{N}\sum_{j=1}^{N}\cos\left(\frac{j}{N}\cdot\frac{\pi}{2}\right)$ is the Riemann sum for

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x \; dx = \frac{2}{\pi} \quad ,$$

and for large N we can approximate $\sin \pi/2N$ by $\pi/2N$, so that

$$\lim_{N\to\infty} \ell_N = \pi\left(1+\frac{2}{\pi}\right) = \pi + 2 \quad .$$

Therefore $\lim_{N\to\infty} \mathcal{H}^1(C_N) = 2+\pi$, but unfortunately the limit object C_∞ in the Hausdorff topology is the unit circle with $\mathcal{H}^1(C_\infty) = 2\pi$. This provides another example that demonstrates the lack of weak lower semicontinuity of $\mathcal{H}^1$ along sequences of curves whose number of components becomes unbounded. However, if you want to find a water pipe (of positive) thickness under a circular garden bed, without digging outside the bed, the above construction of C_N offers a pretty good strategy.

I tried to trace the origins of the opaque square problem and discovered that the opaque disk seemed to be much younger. It is described for instance in [18, p.39-41], where a related problem is also described, namely that of finding one's way out of a forest. More on the forest problem can be found in 35] and [41, 42].

II. Newton's problem of minimal resistance

More than 300 years ago Isaac Newton solved a puzzling variational problem. He tried to minimize the resistance of a rotational body travelling through a liquid, and one of his motivations was that his ivestigations would be of interest to "Her Majesty's Navy". In a place like Troia, where this summer school is held, the importance of building fuel efficient ships is quite obvious and one way to optimize fuel efficiency is undoubtedly to minimize drag. But what exactly is drag? Newton placed himself, so to speak, in the boat and thought of a liquid as a (rarefied) particle stream that flows past the boat or more generally, past a body in $\mathbb{R}^3$. He stated for instance in his *Principia Mathematica* (1686):

"If in a rare medium consisting of equal particles freely disposed at equal distances from each other, a globe and a cylinder described on equal diameters move with equal velocities in the direction of the axis of the cylinder, the resistance of the globe will be but half as great as that of the cylinder."

Today we would briefly say: "The drag coefficient of a ball is 0.5."

Incidentally, the drag coefficient of a TGV or a sportscar is somewhere below 0.3. Drag coefficient is the ratio of the drag of a body B over the drag of a flat plate whose shape is the projection of B in direction of the flow.

Drag itself is measured in absolute terms by the total momentum received from fluid particles to which the body is exposed at any given time. This momentum is caused by collisions of the fluid particle with the body B, and for lack of a more sophisticated model at his time, Newton used Newtonian mechanics. He assumed that the particles do not interact with each other, and that upon impact on the obstacle they are reflected in a perfectly elastic way and disappear to infinity. These assumptions are violated in real gases around us, but they are valid in rarefied gases such as the upper layers of our atmosphere. In fact, in the 1960s, NASA tested the validity of the model when predicting the drag of orbiting rockets, see [47] and the references therein. In order to express drag in terms of the geometry of a body, let us imagine that the direction of flow is vertically downward. Only the upper part of a body (at rest) is then exposed to collisions with the particle stream, and each particle can lose up to 100 percent of its kinetic energy when striking the body.

To fix ideas, let us assume a rotational body B, whose axis of rotation coincides with the direction of the flow, and call its maximal cross section Ω. Any shape of the body below the plane of maximal cross section, does no longer influence the flow (according to this model). The surface of B that lies above Ω is supposed to be parametrized by a function u of r, where r measures the distance of a point x in Ω to its center, i.e. without loss of generality to the origin. Moreover we assume that u is nonnegative in Ω. If Ω is the unit disk D, and B a cylinder in the direction of the flow, then $u \equiv const.$ in D. If B is a ball of radius 1, however, then $u(r) = \sqrt{1-r^2}$, if B is a circular cone with base D and height h, then $u(r) = (1-r)h$ and so on.

Now the path of a particle approaching the body B and its reflection span a vertical plane and are depicted in Figure 2.1. The tangent to u in the point of reflection and the vertical form an angle θ. If we split the momentum vector of the particle into a component tangential and one normal to the tangent of u at the point of impact, then the tangential momentum is conserved, but the normal one is reversed. Trigonometry shows that the proportion of the normal component is $\sin(\theta)$ times the original momentum. Thus the normal component exerts drag, but only its vertical component in the direction of the flow slows the body down. The horizontal component might cause a drift to the side, however, there is a balance for this drift coming from the opposite side of the rotational body, so that they annihilate each other.

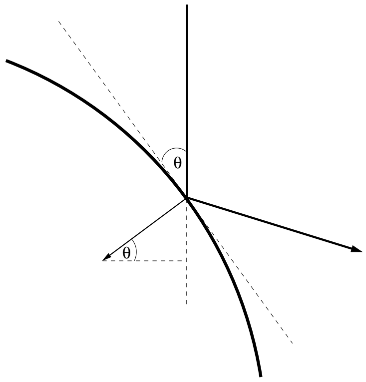

Fig.2.1. The $\sin^2(\theta)$-law of Newton

Thus the drag exerted in vertical direction at a point x is proportional to $\sin^2(\theta)$, that is to $(1+u_r^2)^{-1}$, and the total drag on the body is proprtional to

$$R(u) = \int_D \frac{1}{1+u_r^2}\ dx = 2\pi \int_0^1 \frac{1}{1+(u_r(\rho))^2}\ \rho\ d\rho. \tag{2.1}$$

Using (2.1) as a definition for (absolute) resistance, it is now a simple exercise in integration to verify that the drag coefficient of a ball is 0.5, and the one of a circular cone of height h on the unit disk is $1/(1+h^2)$. The higher and more slender the cone, the smaller is its resistance according to Newton. What is the body of minimal resistance? This is a shape optimization problem par excellence, and Newton offered a surprising solution for it. Recall that he was one of the founders of infinitesimal calculus as well as of the calculus of variations. Thus he did not exactly specify the class of admissible functions u for the functional R. In fact, he had imagined the rotational body of length M as having a horizontal axis of rotation and being exposed to a horizontal flow. Therefore setting $v(t) = u^{-1}(M-t)$ and assuming that $u_r \neq 0$, the functional can be rewritten in the form

$$\int_0^M \frac{vv'^3}{1+v'^2}\ dt, \tag{2.2}$$

in which it shows up in classical books such as [65, Fu, 30]. Rotating v around the horizontal t-axis would generate the surface of B. This is an indication that Newton might have had radially decreasing functions u satisfying $o \leq u \leq M$ as admissible functions in mind when he attacked the problem of finding a body of minimal resistance.

In this class of (formerly unspecified) functions one can seek to minimize R by solving the associated Euler-Lagrange equation. The first variation of R in direction of a test function $\phi(r)$ should vanish at the minimizing u, i.e

$$\int_0^1 \frac{r}{(1+u_r^2)^2}\ 2u_r\phi_r\ dr = -2\int_0^1 \frac{d}{dr}\left(\frac{ru_r}{(1+u_r^2)^2}\right)\ \phi\ dr = 0, \tag{2.3}$$

which implies that

$$\frac{d}{dr}\left(\frac{ru_r}{(1+u_r^2)^2}\right) = 0$$

or

$$h(u_r) := \frac{u_r}{(1+u_r^2)^2} = \frac{const.}{r}. \tag{2.4}$$

Note that the function $s \mapsto h(s)$ is bounded on $\mathbb{R}$, so that the lefthand side of (2.4) is bounded, while the righthand side becomes unbounded as $r \to \infty$. (The case that $const. = 0$ gives only the solution $u \equiv M$ which maximizes R and which is of no interest to us.) Therefore, for $const. \neq 0$ one cannot solve (2.4) in a neighborhood of the origin. Nevertheless Newton offered a body of minimal resistance. The graph of Newton's optimal solution, which I will call u_N from now on, consists of two parts, a constant top $u \equiv M$ on $[0, r_0]$, and a concave arc on $[r_0, 1]$. The arc solves (2.4) for a suitable $const.$, and both r_0 and $const.$ are chosen in such a way that $u_r(r_0^+) = -1$ and $u(1) = 0$. For $M = 1$ the function $u_N(r)$ is sketched in Figure 2.2. Its calculation is a nice exercise in ODE and can be found in [11].

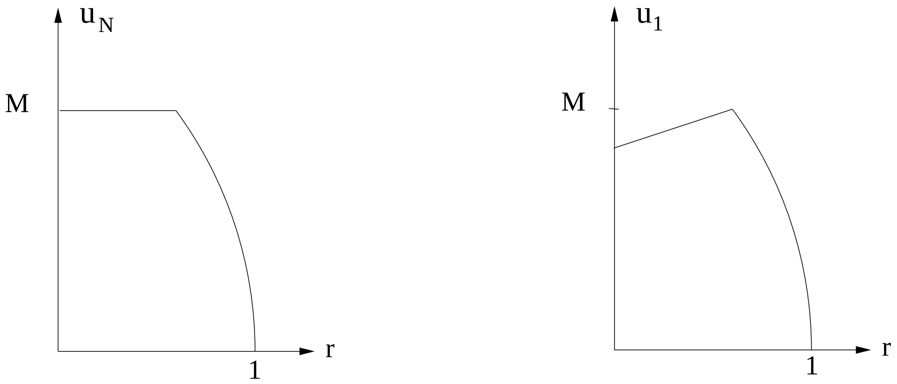

Fig.2.2. Newton's optimal solution u_N and a modification u_1

Rotating u_N about the vertical axis results in something like the shape of a bullet, and everybody who understands a little bit of fluid dynamics will immediately dismiss Newton's solution as not optimal. It has a blunt front end, a set of stagnation points, where fluid particles will be blocked, causing increased pressure and temperature and so on. This reasoning leaves the realm of Newton's physical model of "particles freely disposed at equal distances from each other".

It seemed to be a mystery how Newton had come up with the boundary condition $u_N'(r_0^+) = -1$. D.T.Whiteside has dedicated many years of his life to studying Newton's mathematical papers and has written a series of carefully researched books about Newton's papers and subsequent developments. He has reached the conclusion about the boundary condition, that "This must surely be an educated guess on Newton's part.", see [66, p.478]. Newton did in fact have deep insight, and today one can give a rigorous derivation of this condition, see [8, Thm. 2.3].

Nevertheless it is worthwile to look at Newton's original reasoning, which has been reproduced in [45]. Newton used Figure 2.3 (to which I have added a u and an r-axis) to illustrate his reasoning.

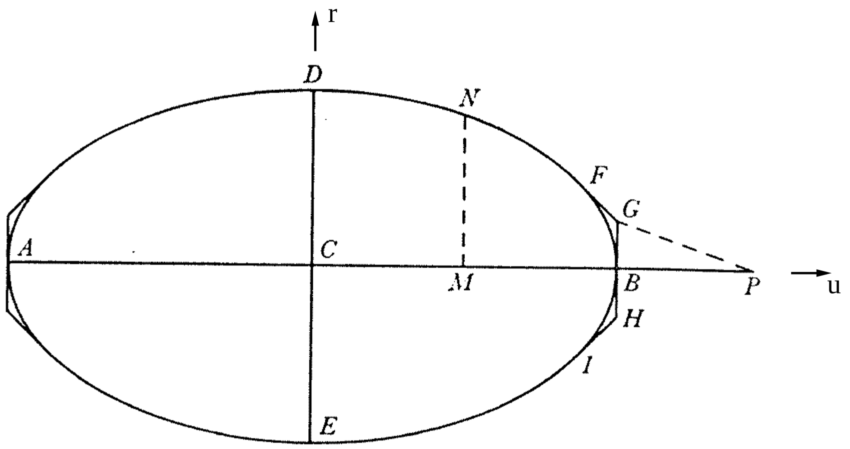

Fig.2.3. Newton's drawing, see [66], p. 462

After deriving his model of resistance, he continues as follows (calling straight lines right lines).

"Incidentally, ..., it follows from the above that, if the solid $ABDE$ be generated by the convolution of an elliptical or oval figure $ADBE$ about its axis AB, and the generating figure be touched by three right lines FG, GH, HI in the points F, B and I, so that GH shall be perpendicular to the axis in the point of contact B, and FG, HI may be inclined to GH in the abgles FGB; BHI of 135 degrees: the solid arising from the convolution of the figure $ADFGHIE$ about the same axis AB will be less resisted than the former solid, provided that both move forwards in the direction of their axis AB, and that the extremity B of each go forward. This Proposition I conceive may be of use in the building of ships."

Let me remark in passing, that the construction of w in the proof of Theorem 3.2 below was inspired by this reasoning of Newton. Newton then goes on to comment on the curved part of his object.

"If the figure $DNFG$ be such a curve, that if, from any point thereof, as N, the perpendicular NM be let fall on the axis AB, and from the given point G there be drawn the right line GP parallel to a right line touching the figure in N, and cutting the axis produced in P, MN becomes to GP as GP^3 to $4BP \cdot GB^2$, the solid desribed by the revolution of this figure about its axis AB, moving in the before.mentioned rare medium from A to B, will be less resisted than any other circular solid whatsoever, described of the same length and breadth."

To interpret this explanation in terms of formulas, I added two axes to Newton's figure, an r-axis vertically upward through D and a u-axis along AP, pointing in that direction. Then B is $(0, M)$ and D is $(1, 0)$, while we denote the length of BG by r_0. Now Newton's statement transforms into current notatation in the following way. Let the optimal curve $BGFND$ joining B to D be the graph $r \mapsto u(r)$, $r \in [0,1]$. Then u is constantly M on $[0, r_0]$, at r_0 its right derivative is -1, and on $(r_0, 1)$ the function u satisfies the equation

$$MN = r = -r_0 \, \frac{[1+u_r^2]^2}{4u_r} = \frac{GP^4}{4BP \cdot GB^2},$$

where we have used that $GP^2 = BG^2 + BP^2$. Notice that this is nothing but (2.4).

Newton's solution was apparently not well understood by his contemporaries, nor by subsequent generations. About a century after Newton's solution, Legendre criticized it. Legendre observed that one could diminish R to as close to zero as one would wish by choosing a function u that has a large derivate everywhere, but oscilllates in order to stay within the L^∞ bound. To find away out of this dilemma, he had to exclude such minimizing functions by restricting the class of admissible ones. Legendre chose to prescribe the one-dimensional arc length as a side constraint. This constraint has no physical meaning, and it leads to an Euler-Lagrange equation different from (2.4), which Legendre was able to solve in special cases. Belloni and I have revisited Legendre's paper in [8], and I refer to it for more details.

Other ways out of Legendre's dilemma have been suggested. Further 2 centuries later McShane argues in [45], that oscillating slopes of u would generate "troughs which fill up with stagnant fluid" so that one obtains "an effective surface giving a nonincreasing u". His reasoning ignores Newton's assumption of noninteracting "particles freely disposed" as well.

The only argument to rule out admissible functions like zig-zag functions with large slope, which is consistent with Newton's physics, is the following: Those functions generate troughs, when rotated, and fluid particles can get caught in the trough in the sense that the hit the body more than once. Their momentum at the second impact, however, is not measured by the functional R. Suppose a particle hits the body first at $(x, u(x))$, then bounces off and hits again at $(y, u(y))$, this time not vertically from above. The functional R catches only the momentum which the point $(y, u(y))$ is exposed to from a particle travelling vertically downward. So a physically correct class of admissible functions seems to be made up of those functions which satisfy the so-called single shock property, that every fluid particle interacts with the body at most once. This is a geometric condition. We want a particle path, that is reflected at $(x, u(x))$ to lie above the rest of the obstacle. In terms of formulas, if $u'(x) < 0$, this translates into (see Fig. 2.4):

$$u(y) \le u(x) + \frac{1}{2}\left(u'(x) - \frac{1}{u'(x)}\right)(y - x) \quad \text{for } y > x. \tag{2.5}$$

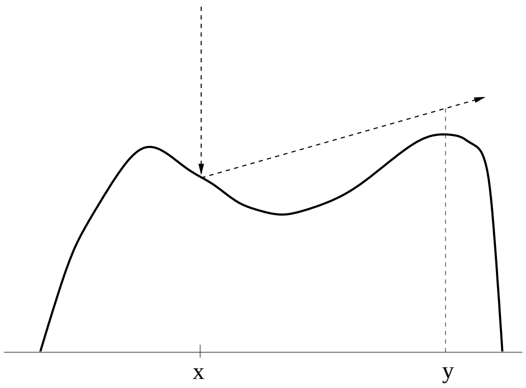

Fig.2.4. Derivation of (2.5)

Let me derive (2.5) by noting that the incoming and outgoing direction v_{in} and v_{out} of the particle can be expressed in terms of the tangential and normal vectors τ and ν to u at x as

$$v_{in} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot \nu\right] \nu + \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot \tau\right] \tau$$

$$\text{and} \qquad v_{out} = - \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot \nu\right] \nu + \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot \tau\right] \tau,$$

because the normal direction of a particle is reversed upon impact. Since

$$\tau = \frac{1}{\sqrt{1+u'(x^2)}} \begin{pmatrix} 1 \\ u'(x) \end{pmatrix} \quad \text{and} \quad \nu = \frac{1}{\sqrt{1+u'(x^2)}} \begin{pmatrix} -u'(x) \\ 1 \end{pmatrix},$$

one calculates

$$v_{out} = \frac{1}{\sqrt{1+u'(x^2)}} \begin{pmatrix} 2u'(x) \\ u'^2(x) - 1 \end{pmatrix},$$

which implies that the path of the reflected particle has a slope of magnitude $\frac{1}{2}(u'(x) - \frac{1}{u'(x)})$. This proves (2.5).

Newton himself spoke of "oval bodies" when he derived his model, and he illustrated it only with convex bodies. Obviously convex bodies have the single shock property, but they form only a subclass. In fact, concave functions u would satisfy the stronger condition

$$u(y) \leq u(x) + u'(x)(y - x). \tag{2.6}$$

An example of a function satisfying (2.5) but not (2.6) is a modification of Newton's optimal solution, which is depicted in Figure 2.2. Take u_N and replace the flat top by a circular cone of opening angle $2\pi/3$. Then every particle that falls into this cone will be reflected by the body and leave the cone in a radial direction parallel to the cone without hitting the body another time. Since this modified solution, u_1 say, has a nonzero gradient on the disk with radius r_0, it has a smaller resistance than Newton's original solution u_N.

Therefore we have now two classes of admissible functions to choose from, namely

$$C_M := \{u \in W^{1,\infty}_{loc}(D), 0 \le u \le M, u \text{ convex}\} \qquad \text{or}$$
$$S_M := \{u \in W^{1,\infty}_{loc}(D), 0 \le u \le M, u \text{ satisfies the single shock condition}\}.$$

As we have just seen, among radial functions the solution of Newton is not optimal in S_M, but in C_M it is. To prove any existence result via the direct method in the calculus of variations we must show that the functional R is lower semicontinuous along a minimizing sequence u_k of functions in the admissible set, be it S_M or C_M. Using (2.5) and (2.6) one can fairly easily show that functions in S_M satisfy

$$|u_r(\rho)| \le \frac{M + \sqrt{M^2 + (1-\rho)^2}}{1-\rho}, \qquad \text{for a.e. } \rho \in [0,1), \tag{2.7}$$

while those in C_M satisfy

$$|u_\rho| \le \frac{M}{1-\rho}, \qquad \text{for a.e. } \rho \in [0,1). \tag{2.8}$$

Therefore minimizing sequences in S_M or C_M are a priori uniformly bounded in $W^{1,\infty}(D')$, where D' is any disc with radius less than 1, centered at the origin, and thus they have weakly convergent subsequences in $W^{1,p}(D')$ for any $p \in [1,\infty)$. This is not enough information to obtain lower semicontinuity along a sequence in S_M. Let me introduce a sequence u_k of functions which is a modification of Newton's u_N. u_1 is defined above, and u_k (as a function of ρ) has a zig-zag shape consisting of k line segments, where u_N used to be flat. Clearly the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded even in $W^{1,\infty}(D)$ and converges strongly in $L^\infty(D)$ and weakly in $W^{1,p}(D)$ to u_N as $k \to \infty$, but $R(u_k) = R(u_1) > R(u_N)$ for any $k \in \mathbb{N}$.

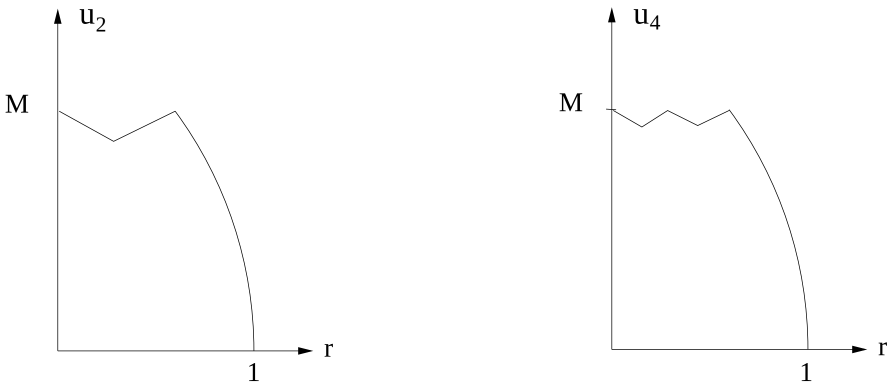

Fig.2.5. Plot of u_2 and u_4

This behaviour should not come as a surprise to us, because R is not convex in $|u_r|$ and therefore not weakly lower semicontinuous in $W^{1,p}(D)$. What is surprising, is the fact that in C_M one can get an existence proof after all, and as far as I can tell, the first rigorous existence proof for radial functions in C_M seems to have appeared only in 1990 [44], three centuries after Newton found more or less the explicit solution in the first place. His proof was later generalized in [12] and will be given below, when I will treat more general (nonrotational) convex bodies. As a matter of fact, for radial functions in C_M one can prove the existence of a minimizer of $R(v)$, and show that Newton's u_N is this uniquely determined minimizer, while in S_M it is not easy to even guess an optimal u. The picture will change, though, if nonradial functions are also admitted for variations. This will be done in Chapter III, below.

In Figure 2.5 I have sketched the qualitative behavior of a sequence of functions in S_M. This sequence oscillates on the interval $(0, r_0)$ in a regular zig-zag fashion with slope $\pm 1/\sqrt{3}$, and for $M = 1$ we had picked $r_0 = .35$ as in Newton's solution. If we do variations on r_0, then we need another curved arc reaching from (r_0, M) down to $(1, 0)$. Such arcs can be constructed from the Euler equation (2.4) or equivalently from solutions of Newton type with different M. This construction was done in great numerical detail in [62] for $M = 1$, and D.Stolz figured out the following numerical values:

- $R(u_N) = \ 1.163$
- $R(u_k) = \ 1.073 \quad \text{if } r_0 = .35 \quad \text{as for } u_N$
- $R(u_k) = \ 1.022 \quad \text{if } r_0 = .636$
- $R(u_k) = \ 1.378 \quad \text{if } r_0 = .765$

These values suggest that for $M = 1$, it pays to enlarge r_0 over Newton's $r_0 = .35$ and to drill a cone or a couple of concentric troughs of slope $\pm 1/\sqrt{3}$ into the blunt part, to reduce drag. Somehow such a roughening of surfaces reminds us of the dimples in golf balls or of the fact that rough surfaces are known to reduce the drag coefficients of ships and airplanes. These real world phenomena are based on other effects, though, such as the break up of turbulence (Divide et impera!).

There is another conceivable way to improve Newton's solution. Note that the particles coming out of the cone come out parallel to the cone. Can one construct a radial valley of radius r_0 such that every incoming particle just about makes it across the ridge on the opposite side? This is a question from geometric optic. To construct such a valley one has to investigate the equality case in (2.5), see also Figure 2.5.

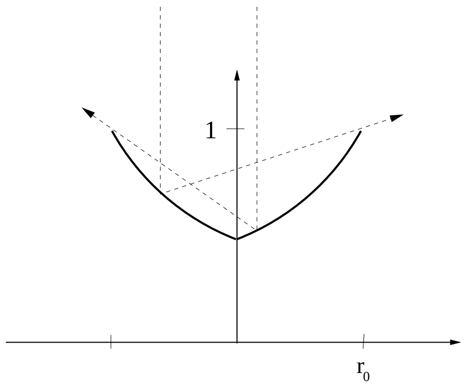

Fig.2.6. Shape of a function satisfying equality in (2.5)

The function depicted in Figure 2.6 is a piecewise polynomial of degree 2, with slope 1 at the outer edge r_0 and slope $1/2$ at the origin. Note that the slope of the cone was $\sqrt{3}/2$ which is between 1 and $1/2$. So which "valley" has less resistance, the conical one or the parabolic one? And does the resistance of a body change if one carves many concentric parabolic troughs into it instead of one big valley as in Figure 2.7?

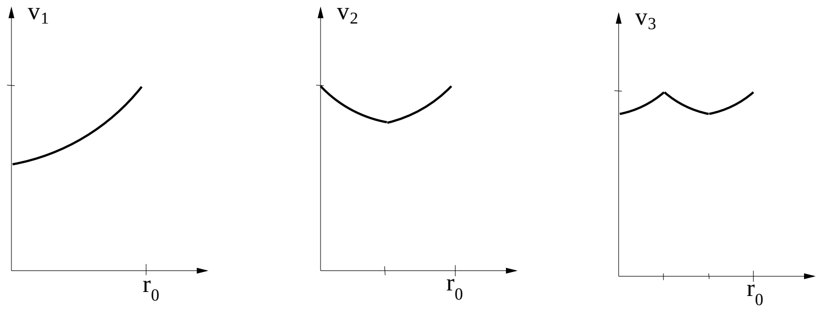

Fig.2.7. Parts of the Graphs of v_1, v_2, v_3

Only a calculation can show. This question, too, was investigated by D.Stolz. She compared the resistance of the blunt part of Newtons body with the one of the modified parts, i. e. she considered

$$\overline{R}(v) = 2\pi \int_0^{r_0} \frac{1}{1 + (v_r(\rho))^2}\ \rho\ d\rho, \tag{2.9}$$

and computed that

$$\begin{aligned} \overline{R}(u_N) &= 3.142\ r_0^2, \\ \overline{R}(v_1) &= 1.854\ r_0^2, \\ \overline{R}(v_2) &= 2.020\ r_0^2 = \overline{R}(v_k) \text{ for } k \text{ even}, \end{aligned}$$

and

$$\overline{R}(v_k) = \frac{4\pi}{k^2}\left[\ln\frac{8}{5} + \left(\frac{k^2-3}{2}\right)\left(\frac{\pi}{4} + \arctan\left(-\frac{1}{2}\right)\right)\right]\ r_0^2$$

for k odd. Note that in this latter case $\overline{R}(v_k)$ increases in k as $k \to \infty$ to $\overline{R}(v_2) = 2\pi(\frac{\pi}{4} + \arctan(-\frac{1}{2}))r_0^2$, while $v_k \to u_N$ in L^∞ – another illustration for lack of lower semicontinuity. Therefore among the functions v_k a single valley, i.e. v_1, is optimal. Now for the cone u_1 it is easily seen that resistance is inbetween, because

$$\overline{R}(u_1) = \frac{3}{4}\pi r_0^2 = 2,355\ r_0^2.$$

Summarizing these results, it is a good strategy to carve piecewise parabolic troughs into $u_N(r)$ and to vary r_0 when one wants to minimize $R(v)$ in the class of radial functions in S_M. The minimizer there – if it exists – still remains to be found.

III. More on Newton's problem

In this lecture I shall modify Newton's original problem in several ways. First of all, I allow for nonrotational bodies, so that the admissible functions for R are no longer radial. Instead of being defined on a disk they are defined on an arbitrary domain $\Omega \subset \mathbb{R}^N$. Now the functional reads

$$R(v) := \int_\Omega \frac{1}{1+|\nabla v|^2}\ dx, \tag{3.1}$$

and we seek to minimize it in the following class of admissible functions:

$$C_M := \{\ v \in W^{1,\infty}_{loc}(\Omega), 0 \le v \le M, v\ \ \text{convex}\ \}.$$

The following can be shown.

Theorem 3.1:
Let $\Omega \subset \mathbb{R}^N$ be convex.
a) There exists a minimizer u of R in C_M.
b) The minimizer is in general not unique, because if $\Omega = D$, the unit disk in $\mathbb{R}^2$, u is not radial.
c) The Euler-Lagrange equation associated with R is formally

$$-\text{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^2}\right) = 0; \tag{3.2}$$

it is of mixed (elliptic-hyperbolic) type, and holds in the interior of C_M, i.e. in points where u is strictly concave and where $u(x)$ lies strictly between 0 and M.

Proof of a): Since R is nonnegative, it has an infimum on C_M. Let u_k with $k \in \mathbb{N}$ be a minimizing sequence for R in C_M. What can be said about convergence of this sequence? For many coercive variational problems one can derive convergence from the fact that minimizing sequences are bounded in some norm. Our functional is not coercive, in fact a minimizer will try to have a large gradient, because this will decrease the integrand. In this problem we show that compactness of minimizing sequences comes from the suitable choice of admissible functions. They are compact in the strong topology of $W^{1,p}_{loc}(\Omega)$ for any $p \in [1,\infty)$. To realize this, one has to first observe that functions in C_M satisfy the inequality (which for radial functions degenerates to (2.8)

$$|\nabla v(x)| \leq \frac{M}{dist(x,\Omega)} \qquad \text{a.e. in } \Omega. \tag{3.3}$$

This can be seen as follows. For almost every point $x \in \Omega$ the function v is differentiable, and the normal vector to the graph of v together with the "vertical"unit vector span a plane, restricted to which we can plot the graph of v as a convex, one–dimensional function.

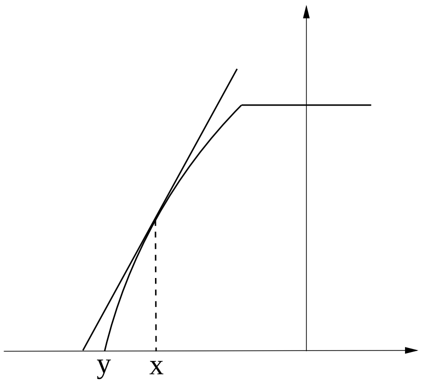

Fig.3.1. Derivation of (3.3).

Its tangent in x has to cross $\partial\Omega$ above height zero, hence we obtain the chain of inequalities $0 \leq v(y) \leq v(x) + (y-x)v'(x) \leq M - |y-x||v'(x)|$ for every boundary point y in this plane. Note from an inspection of Figure 3.1, that $(y-x)v'(x) \leq 0$. Therefore $|v'(x)| \leq M/|y-x|$, and the right hand side in this inequality becomes larger if I decrease the denominator to $dist(x,\partial\Omega)$. This establishes (3.3). Therefore any sequence u_k in C_M is locally uniformly Lipschitz-continuous, i.e. for every compact $\Omega' \subset\subset \Omega$ it is uniformly Lipschitz-continuous on Ω', and, due to the Arzela-Ascoli Theorem, after possibly passing to a subsequence, it has a locally uniform concave limit u. This shows that C_M is compact in $L^\infty_{loc}(\Omega)$. Let us now show that moreover

$$\nabla u_k(x) \to \nabla u(x) \qquad \text{a.e. in } \Omega. \tag{3.4}$$

Then (3.4), (3.3) and the Lebesgue Dominated Convergence Theorem imply compactness of C_M in every $W^{1,p}_{loc}(\Omega)$. To verify (3.4) let x be a point where all the u_k and u are differentiable, which is almost everywhere in Ω. By concavity of the one-dimensional functions $t \mapsto u(x+te_i)$ (with e_i denoting the i-th unit vector in $\mathbb{R}^N$ for $i=1,2,\dots,N$) the derivative of this function at x can be bounded below and above by the difference quotient from the right and from the left, i.e.

$$\frac{u_k(x+\varepsilon e_i)-u_k(x)}{\varepsilon} \le \frac{\partial}{\partial x_i}u_k(x) \le \frac{u_k(x-\varepsilon e_i)-u_k(x)}{-\varepsilon} \tag{3.5}$$

holds for a.e. $x \in \Omega$, for every $k \in \mathbb{N}$ and every $i=1,2,\dots,N$. There are two limits $k \to \infty$ and $\varepsilon \to 0$ that we can apply to (3.5). Passing with k to infinity first, gives

$$\begin{aligned}\frac{u(x+\varepsilon e_i)-u(x)}{\varepsilon} &\le \liminf_{k\to\infty}\frac{\partial}{\partial x_i}u_k(x)\\ &\le \limsup_{k\to\infty}\frac{\partial}{\partial x_i}u_k(x) \le \frac{u(x-\varepsilon e_i)-u(x)}{-\varepsilon}\ ,\end{aligned} \tag{3.6}$$

and sending ε to zero in (3.6) yields the proof of (3.4), namely

$$\frac{\partial}{\partial x_i}u(x) \le \liminf_{k\to\infty}\frac{\partial}{\partial x_i}u_k(x) \le \limsup_{k\to\infty}\frac{\partial}{\partial x_i}u_k(x) \le \frac{\partial}{\partial x_i}u(x) \quad \text{a.e. in} \quad \Omega.$$

Now (3.4) and Lebesgue's Dominated Convergence Theorem imply that $R(u) \le \liminf_{k\to\infty} R(u_k)$ and prove a).

Before proceeding with the proof of b) let me *remark*, that for v in the class S_M one can also find a locally uniform gradient bound which reminds us of (2.7)

$$|\nabla v(x)| \le \frac{M+\sqrt{M^2+dist^2(x,\partial\Omega)}}{dist(x,\partial\Omega)} \qquad \text{a.e. in } \Omega,$$

a circumstance, which might raise hopes of being able to prove existence of minimizers in S_M as well. But as the examples of sequences in my second lecture have shown, in general (3.4) is violated for sequences in S_M. Therefore I do not know how to get a handle on the existence problem in S_M, not to mention uniqueness and other properties that I can investigate in the C_M case.

Proof of b): One can give several proofs for the nonsymmetry. The third one (chronologically) that was found consists in showing that for a peculiar nonradial function $v \in C_M$ we have $R(v) < R(u_N)$ so that the minimum over radial functions in C_M, which is attained at Newton's solution u_N, lies above the minimum in general functions in C_M. This special function v looks like a lipstick or screwdriver and was found by P.Guasoni [31]. A picture of it can also be found in [25]. For a long time I had tried to prove the converse, that a minimizer in C_M had to be radial if $\Omega = D$ was a circular disk, see [12, p.85-86]. Then my coauthors and I followed a suggestion of H.Brezis and looked at the second variation of R at u_N. If a test function $\phi \in C^\infty(D)$ has the property that for every ε in a sufficiently small neighborhood of zero the function $u_N + \varepsilon\phi)$ is an element of C_M, and if u_N is a global minimizer of R in C_M, then $R(u_N + \varepsilon\phi)$ can be interpreted as a function of ε, which has a local minimum at $\varepsilon = 0$. Consequently the first variation $\delta R(u_N, \phi)$ in direction of this testfunction vanishes, i.e.

$$\delta R(u_N, \phi) = -2 \int_D \frac{\nabla u_N \cdot \nabla \phi}{(1 + |\nabla u_N|^2)^2}\, dx = 0, \tag{3.7}$$

and its second variation $\delta^2 R(u_N, \phi)$ is nonnegative, i.e.

$$\delta^2 R(u_N, \phi) = \int_D \frac{2\left[-(1 + |\nabla u_N|^2)|\nabla\phi|^2 + 4(\nabla u_N \cdot \nabla\phi)^2\right]}{(1 + |\nabla u_N|^2)^3}\, dx \geq 0. \tag{3.8}$$

We intend to show that (3.8) can be falsified. This implies that u_N is not a minimizer in C_M. but first let us note that as long as we stay with ϕ in the class of radial functions, the terms $(\nabla u_N \cdot \nabla\phi)^2$ and $|\nabla u_N|^2 |\nabla\phi|^2$ coincide and are equal to $u_r^2\ \phi_r^2$. Therefore in the radial case the square bracket in (3.8) becomes $(3u_r^2 - 1)\phi_r^2$, and this is positive, because any ϕ such that $u_N \pm \varepsilon\phi \in C_M$ must have its support outside the blunt part of u_N, i.e. in a region where $|\nabla u_N| \geq 1$. Incidentally, this together with c) proves optimality of u_N in the class of radial functions in C_M.

So in order to violate (3.8) we have to choose a function ϕ which is not radial. This can be achieved if the angle between the vectors ∇u_N and $\nabla\phi$ is mostly close to a right angle. So let $\phi(r, \theta) = \eta(r) sin(k\theta)$, where η has its support in $(r_0, 1)$, then the negative term in the square bracket of (3.8) wins over the positive term. This shows that the Newton's radial solution u_N can be infinitesimally perturbed into a function with smaller resistance. In fact, the second proof of nonsymmetry was based on constructing a function close to u_N, and it can be found in [10]. The (chronologically) first proof is the one that falsifies (3.8).

Proof of c): The Euler–Lagrange equation (3.2) follows from (3.7) and the fundamental lemma in the calculus of variations, provided $u \pm \varepsilon\phi \in C_M$ for any sufficiently small nonnegative ε. A sufficient criterion for this is that u is piecewise of class C^2, say $u \in C^2(\Omega')$ with $\Omega' \subset \Omega \setminus (u^{-1}(M) \cup u^{-1}(0))$, and that u is strictly concave in Ω', in the sense that its Hessian is strictly negative, bounded away from zero. In that case any $\phi \in C_0^\infty(\Omega)$ is an admissible variation, because $u \pm \varepsilon\phi$ will be in C_M for sufficiently small ε. It remains to show that (3.2) is of mixed type. To this end we differentiate out and obtain

$$(1 + |\nabla u|^2)^{-3}\ \left[\ (1 + |\nabla u|^2)\Delta u\ - 4\ u_i u_{ij} u_j\ \right] = 0. \tag{3.9}$$

Now recall that the Laplace-operator can be rewritten in any orthogonal coordinate system, for instance in the one spanned by $\nu := -\nabla u/|\nabla u|$ and

τ_i, where τ_i, $i = 1, \ldots, N-1$, denote an orthogonal system of vectors tangent to the level surface at u through x. Then

$$\Delta u = u_{\nu\nu} + \sum_{i=1}^{N-1} u_{\tau_i \tau_i} =: u_{\nu\nu} + u_{\tau\tau} \ ,$$

and the square bracket in (3.9) becomes

$$\begin{aligned} &(1+|\nabla u|^2)\,(u_{\nu\nu}+u_{\tau\tau}) \ - 4\ u_i u_{ij} u_j \\ &= (1-3|u_\nu|^2)\ u_{\nu\nu} + (1+|u_v|^2)\ u_{\tau\tau} = 0. \end{aligned} \tag{3.10}$$

Note that the coefficient in front of $u_{\nu\nu}$ can change sign. If $|\nabla u| < 1/\sqrt{3}$, then it is positive and the equation is elliptic, while $|\nabla u| > 1/\sqrt{3}$ yields a negative coefficient and a hyperbolic equation. This completes the proof of Theorem 3.1.

Remark 3.1:
Newton's solution, and as we shall see in the next Theorem, minimizers in C_M as well, have $|\nabla u| \geq 1$ in the interesting parts of their graph. Does this mean that they solve a hyperbolic equation there? Yes, but only if the assumptions for the derivation of (3.2) are satisfied. The latter seems unlikely and one could also argue that a minimizer u should satisfy the Monge-Ampère equation

$$\det D^2 u = 0 \quad \text{a.e. in } \Omega' \tag{3.11}$$

instead of (3.2). This was pointed out to me by H. Berestycki. The (heuristic) argument goes as follows. Suppose u minimizes R in C_M and is strictly concave and of class $C^2(\Omega')$ where Ω' is a subset of Ω in which u is neither M nor 0. Then, as in the proof of the nonsymmetry part of Theorem 3.1 above, we could construct an admissible testfunction which is highly oscillatory in directions orthogonal to ∇u, and for which the second variation of R at u becomes negative. This contradiction implies that u is not smooth or not strictly concave. In the latter case, because it is still concave, one of its principle curvatures must vanish, i.e. the Hessian has vanishing determinant.

So which equation, (3.2) or (3.11) is the right one for minimizers? I do not know. There is numerical evidence which supports (3.11), but maybe neither one is correct. It would be instructive to know what the minimizer in C_M looks like if D is a disk, but even this special question is still open.

Other things are known, however. In the following Theorem 3.2 I want to give a mathematical justification for Newton's "educated guess"(see Chap. II above) that a minimizer has always slope outside the interval $(0,1)$.

Theorem 3.2:
Let $\Omega \subset \mathbb{R}^N$ be convex and u be a minimizer of R in C_M. Then $|\nabla u| \notin (0,1)$ a.e. in Ω.

For the proof we argue by contradiction and show that u cannot be minimizer, if $|\nabla u(x)| \in (0,1)$ on a subset, N say, of Ω with positive measure. In this case we replace u by a function $w \in C_M$ which lies above u, and has slope outside $(0,1)$. Following the idea of Newton in Figure 2.3 above, such a function can be constructed by

$$w(x) := \inf\{\ t(x)\ :\ t(x) \text{ is a tangent plane to } u \text{ with slope } \notin (0,1)\ \},$$

and because of the concavity of u it touches u in those points where $|\nabla u| = 0$ or $|\nabla u| > 0$. To realize why $R(w) < R(u)$, it helps to "relax" R and to define its relaxation $\tilde{R} = \int_\Omega \tilde{f}(|\nabla u|)\,dx$, by replacing the map $f(s) = 1/(1+s^2)$ with something more convex, namely $\tilde{f}$. The largest convex function below f in $\mathbb{R}$ is identically zero. Therefore this function is not suitable for our purposes, but if we conceive of f as a function on $[0,\infty)$ and replace this function by its convex envelope, then

$$\tilde{f}s) = \begin{cases} 1 - s/2 & \text{if } s \in [0,1] \\ 1/(1+s^2) & \text{if } s \in [1, 1\infty) \end{cases} \tag{3.12}$$

will do the job. Notice that $f < \tilde{f}$ only in $(0,1)$, so that $\tilde{R}(w) = R(w)$ by construction of w, while $\tilde{R}(u) < R(u)$ by the assumption that N is not a nullset. If we manage to show that

$$\tilde{R}(u) \geq \tilde{R}(w), \tag{3.13}$$

then we have finished the proof. Therefore let us subtract both sides in (3.13) from each other and note that the integrands coincide when $|\nabla u| \notin (0,1)$ to arrive at

$$2\tilde{R}(u) - 2\tilde{R}(v) = \int_{\{u \neq v\}} (|\nabla v| - |\nabla u|)\ dx = \int_\Omega (|\nabla v| - |\nabla u|)\ dx. \tag{3.14}$$

By the coarea formula we have

$$\int_\Omega (|\nabla v| - |\nabla u|)\ dx = \int_0^M \left[\mathcal{H}^{N-1}(\{v = t\}) - \mathcal{H}^{N-1}(\{u = t\})\right]\ dt, \tag{3.15}$$

and the integrand in (3.14) is nonnegative for every $t \in [0, M]$, because the level sets of u are convex subsets of the convex level sets of v, for details of this last statement see [12, p.75]. Therefore the right hand sides of (3.15) and (3.14) are nonnegative, and the proof of Theorem 3.2 is complete.

Remark 3.2:
Other results can be shown for minimizers. I list them here without proof. The natural boundary condition asociated with the functional R is the vanishing flux condition

$$\frac{1}{(1+|\nabla u|^2)^2}\ \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \tag{3.16}$$

but like the Euler equation, it can only be derived under strong assumptions on u. It is also conceivable that $u = 0$ or $u = M$ on the boundary or that u is not strictly concave on $\partial\Omega$. As a matter of fact, u vanishes in at least one boundary point, because otherwise one could stretch the graph of u in vertical direction and obtain a new function in C_M with smaller resistance than u. Even (3.16) can be realized in more than one way, because it is implied by $u_\nu = 0$ or $|\nabla u| = \infty$.

This leads in a natural way to the question if derivatives of u are a priori bounded. In a way they are, see (3.3), but (3.3) breaks down at the boundary. Once we know that $|\nabla u|$ is globally bounded, we also know that $u = 0$ on $\partial\Omega$. To see this, it suffices to do a small variation in the one-dimensional situation depicted in Figure 3.2.

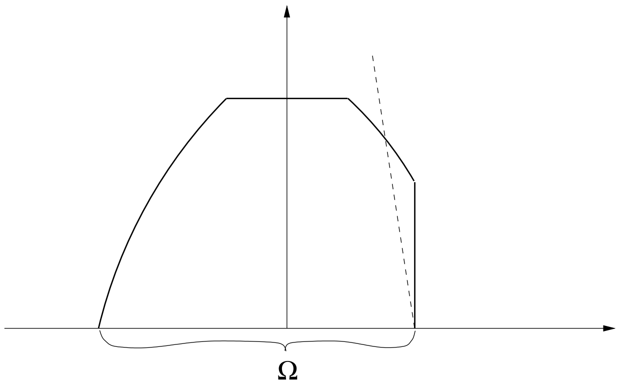

Fig.3.2. Derivation of Dirichlet condition for Lipschitz minimizers.

Replace $u(x)$ by $v(x) = \min\{\ u(x), dist(x, \partial\Omega)/\varepsilon\ \}$ where $1/\varepsilon$ is greater than the bound on ∇u. Then $R(v) < R(u)$, provided $v \neq u$. Otherwise $v = u = 0$ on $\partial\Omega$.

Aside from Aleksandrov's Theorem about C^2 differentiability a.e. for concave functions and thus for every admissible function, no additional regularity result seems to be known for minimizers of R in C_M. Incidentally, under moderate regularity assumptions, we can improve on Theorem 3.1b) and show that $|\nabla u|$ jumps precisely from 0 to 1 on the boundary of the blunt part $\partial\{u = M\}$, see [12, p.87].

What about numerical calculations of the minimizer? The functional and the admissible set of functions are somewhat nonstandard. R is neither convex nor coercive, and concavity is a pointwise constraint on second order derivatives, or a nonlocal constraint, namely

$$u(tx + (1-t)y) \geq tu(x) + (1-t)u(y) \qquad \text{for any } x, y \in \omega,\ t \in [0,1] \quad (3.17)$$

on the admissible functions themselves. If one wants to do a finite element approximation of a minimizer with piecewise linear finite elements, these are not of class C^2 across their "edges". Therefore we have to incorporate

(3.17) into a numerical formulation of the problem. I report now on joint work with C.Schwab [40]. To simplify matters let Ω be the unit square. We triangulate it in a regular way with triangles of gridsize h. Instead of looking for a function of $x \in \Omega$ we look now for a vector in $\mathbb{R}^N$ with $N = O(h^{-2})$, whose components should be u evaluated at each gridpoint. The functional R can be discretized by splitting the integral into a sum of integrals over each triangle. On a given triangle, the gradient of u can be approximated by the affine function that takes the values of u in the gridpoints. This way we can define a discretized version R_h of R, and minimizing R_h constitutes an optimization problem in a finite dimensional space $\mathbb{R}^N$. For the latter ones there are standard optimization algorithms, for instance gradient methods, in which one starts at some point v, evaluates the gradient of R_h at v and modifies v into the direction of steepest descent of R_h. With these concepts in mind it looks fairly easy to do a finite element analysis.

However, the following issues have to be addressed.

a) The discretized functional is not convex either, and therefore any descent method may end in a local, but not global minimizer of the functional. This shortcoming can realistically only be dealt with, if one starts with a good guess in a neighborhood of the optimal solution. Schwab and I took various parabolas with varying eccentricity as initial guesses, and always ended at the “same” minimizer. This gave us some confidence that we were on the right track.

b) Any decent descent algorithm applied to the discretized functional should endeavour to increase the gradient and decrease the functional. It will therefore suggest an increase of values of v in individual gridpoints above M or a decrease below 0, in which case the piecewise linear interpolation would violate the L^∞ bound in C_M. This problem can be dealt with by a simple cut-off at M resp. 0 of the new iterate suggested by the descent algorithm.

c) Finally we may start out with a polygon satisfying the concavity constraint in C_M and obtain a next iteration that is no longer concave. In this case doing local adjustments of the new iterate is not of much help, because (3.17) is a global constraint in Ω. Concavity of minimizers had to be enforced in a global way, and we found a penalty method to be the appropriate tool.

Let me elaborate on this aspect in more detail. As long as a piecewise linear function is concave, it bends the right way, or its gradient jumps in the right direction along the edges of the triangulation. Suppose we measure, how much a (nonconcave) function violates the concavity constraint with some functional $\mathcal{C}_h$, which is zero for concave functions and positive for nonconcave ones, then instead of minimizing R_h we might consider minimizing

$$R_h(v) + \frac{1}{\varepsilon}\mathcal{C}_h(v) \tag{3.18}$$

on the set S_h of all those continuous functions, which are affine on each triangle and take values in $[0, M]$. If $\varepsilon > 0$ is then chosen small enough, the minimizers of (3.18) should be almost concave. This idea works. Let T_k and

T_l be two adjacent triangles with a normal e_{kl} to their common edge pointing from T_k into T_l. Then a piecewise linear u is concave across this edge, provided the scalar product $(\nabla u|_{T_l} - \nabla u|_{T_l}) \cdot e_{kl}$ is nonpositive. Therefore a good way to define $\mathcal{C}_h$ is to measure the positive part of $(\nabla u|_{T_l} - \nabla u|_{T_l}) \cdot e_{kl}$ for every edge in the triangulation and to add it up in some l^s-norm, as in

$$\mathcal{C}_h(v) := \sum_{kl} \left| [(\nabla v|_{T_l} - \nabla v|_{T_l}) \cdot e_{kl}]^+ \right|^s . \tag{3.19}$$

With these ideas fixed let us denote a minimizer of (3.18) by u_ε^h and let us call it a finite element solution. The the following facts can be shown.

Theorem 3.3:
The function R_h is in $C^\infty([0,M]^N)$ and the function $\mathcal{C}_h$ is in $C^{\tilde{s},s-\tilde{s}}([0,M]^N)$, where the functions R_h and $\mathcal{C}_h$ are interpreted as functions of the nodal variables, and where $\tilde{s}$ is the largest integer strictly less than s.
For every $\varepsilon > 0$ and every triangulation of width h there exists a finite element solution u_ε^h.
The finite element solution is almost concave, in the sense that

$$\begin{aligned} u_\varepsilon^h(y) \leq & u_\varepsilon^h(x) + (y-x) \cdot \nabla u_\varepsilon^h(x) \\ & + c\ \varepsilon^{1/s}\ h^{(1-s)/s}\ |y-x| \end{aligned} \tag{3.18}$$

with c independent of $x, y \in \Omega$, h and ε.
If ε satisfies

$$\varepsilon \leq \begin{cases} ch^{1/\alpha} \text{ with } \alpha \in (0, 1/(s-1)) & \text{for } s > 1, \\ o(1) & \text{for } s = 1, \end{cases}$$

then, as $h \to 0$, there is a sequence of finite element solutions, which converges to a limit $u \in C_M$. Moreover, this limit u is a minimizer for the continuous problem.

A proof of Theorem 3.3 is given in [40]. Let me just stress the fact that concave functions satisfy (3.18) with $c = 0$, and that a balance between h and ε is needed so that the last term in (3.18) becomes small as ε and h go to zero.

You may wonder what kind of numerical solution this method delivers. Our numerical caculations led to an optimal u (on a square), the graph of which was a piecewise developable surface, consisting of a square-shaped blunt end and four pieces that seemed to depend only on the distance of x to the boundary. For figures I refer to [40]. Incidentally, these calcuations seem to support the validity of the Monge-Ampère equation (3.11) above. However, for $M = 2$ Paolo Guasoni was able to give an ad hoc example of a polygonal, concave, roof-shaped function, whose resistance was lower than the one of our numerical example. This suggests that we started the descent algorithm with the wrong initial guesses in the first place. So no matter what we try, Newton's problem of minimal resistance seems to resist attack after attack.

To conclude this lecture let me mention a few other aspects of the problem. Presently an existence proof in S_M seems out of reach, but one can restrict S_M a bit an try to look for bodies of fairly small resistance there. In his Diplomathesis M.Mester took cones and carved radial troughs into them, obtaining shapes such as the one depicted in Figure 3.3.

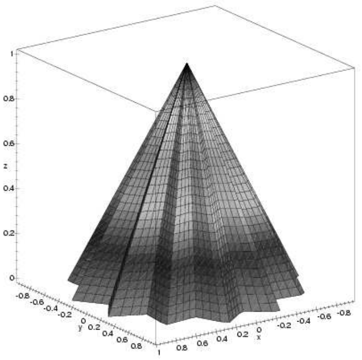

Fig.3.3. Modification of a circular cone.

As the number of creases goes to ∞, they can be less and less deep without violating their membership in S_M. This way one can construct a sequence $v_n \in S_M$ with $\lim_{n\to\infty} v_n(x) = v_\infty(x) = 1 - |x|$ and $R(v_\infty) = 0.5\pi > \liminf_{n\to\infty} R(v_n) = 3\pi/7$, providing yet another case of lacking lower semi-continuity.

What about existence proofs in other classes of admissible functions for slightly modified functionals, which at the same time account for frictional effects? Instead of being concave and bounded in L^∞, it was suggested in 20, p.259] (for the case of radial functions) to limit the surface area of the admissible bodies. In this case Eggers was able to conclude that a minimizer of R had to have conical shape. In [25] Ferone and I looked at this optimal shape problem for a base domain $\Omega \subset \mathbb{R}^N$ which is not necessarily a disk or a ball, but simply connected and bounded.

We set

$$A(S) := \{v \in BV(\Omega) \mid v \geq 0 \ \text{ a.e. in } \Omega \ \text{ and } \ \int_\Omega \sqrt{1 + |\nabla v(x)|^2}\, dx = S\ \}. \tag{3.19}$$

and look for a minimizer of R on $A(S)$.

Theorem 3.4:
For every $S \geq |\Omega|$ there exists a minimizer u of R on $A(S)$. Moreover the function $u(x) = c\ dist(x, \partial\Omega)$ with $c = \sqrt{(s/|\Omega|)^2 - 1}$ is a solution.

To prove this theorem, observe that $S \geq |\Omega|$ is needed to make A nonempty. Now let $v(x)$ be any function from A and define $f(x) := \sqrt{1 + |\nabla v(x)|^2}$. If we replace f by its average

$$\underline{f} = \frac{1}{|\Omega|} \int_\Omega f(x)\ dx = \frac{s}{|\Omega|}, \tag{3.20}$$

we can construct a new function $\tilde{v}$ by solving the following Hamilton Jacobi equation

$$|\nabla \tilde{v}(x)| - \sqrt{\underline{f}^2 - 1} = 0 \quad \text{in } \Omega. \tag{3.21}$$

This will not affect the surface area constraint, because $\int f\ dx = \int \underline{f}\ dx$. Certainly, there exists a nonnegative solution of (3.21), namely a positive multiple of the distance function $dist(x, \partial\Omega)$. Let $\tilde{v}$ be this function, then $\tilde{v} \in A$ as well.

Now we observe that $R(v) = \int f(x)^{-2}\ dx$, that $f \geq 1$ and that the mapping $s \mapsto s^{-2}$ is convex for $s > 0$. Therefore

$$\begin{aligned} R(v) = \int_\Omega \frac{1}{1 + |\nabla v(x)|^2}\ dx &= \int_\Omega \frac{1}{f(x)^2}\ dx \\ &\geq \int_\Omega \frac{1}{\underline{f}^2}\ dx = \int_\Omega \frac{1}{1 + |\nabla \tilde{v}(x)|^2}\ dx = R(\tilde{v}). \end{aligned} \tag{3.22}$$

Thus replacing any function $v \in A$ by a concave function of same surface area but constant slope will not increase R. As a matter of fact, unless v satisfies (3.21), $R(\tilde{v}) < R(v)$.

Remark 3.3:
In [50] F.Nicolai has studied the related problem of minimizing R on

$$\begin{aligned} C(S) := \{ v \in W^{1,\infty}_{loc}(\Omega) \mid v \geq 0, \quad &v \text{ concave in } \Omega \\ \text{and } \int_\Omega \sqrt{1 + |\nabla v(x)|^2}\ dx &+ \int_{\partial\Omega} v\ d\mathcal{H}^{n-1} \leq S \ \}. \end{aligned}$$

She gave an existence proof along the lines of Theorem 3.1 above. In our proof of Theorem 3.4 we do not need the concavity of minimizers. However, for convex Ω our minimizer $c\ d(x, \partial\Omega)$ is also a minimizer of R on $C(S)$.

Note also that again the minimizers of our problem need not be unique. To see this one can look at other solutions of (3.21) under homogeneous Dirichlet conditions. In fact, homogeneous Dirichlet conditions are necessary for Lipschitz continuous minimizers, recall Figure 3.2 above. If we want to somehow identify a unique solution we can ask for a viscosity solution of (3.21) or for a concave solution of (3.21) under vanishing boundary values.

The function $dist(x, \partial\Omega)$, which we have identified as a solution is piecewise of class $C^{1,1}(\Omega)$ and its derivatives degenerate on a subset of Ω which is referred to in 26] as *ridge*. Some examples of domains Ω and their ridges are depicted in 26, p. 197].

Of course Theorem 3.4 remains true if the functional R is replaced by $J(v) = \int_\Omega j(\sqrt{1+|\nabla v(x)|^2})\ dx$ with convex $j : \mathbb{R}^+ \to \mathbb{R}^+$.

Let us finally imagine that the resistance of a body depends also on frictional effects, which were not included in Newton's original model. For simplicity we suppose that the resistance caused by friction is proportional to the surface area of its front and lateral surface. In this case a modified resistance functional would read as follows. For some constant $\alpha > 0$ and for $v \in W^{1,\infty}_{loc}(\Omega)$

$$R_\alpha(v) := \int_\Omega \frac{1}{1+|\nabla v(x)|^2}\ dx\ +\ \alpha \int_\Omega \sqrt{1+|\nabla v(x)|^2}\ dx + \int_{\partial\Omega} v\ d\mathcal{H}^{n-1}. \tag{3.23}$$

Theorem 3.5:
For every $\alpha > 0$ there exists a minimizer u of R on $B := W^{1,\infty}_{loc}(\Omega)$. Moreover, for $a \in (0,2)$ the function $u(x) = \underline{c}\ d(x,\partial\Omega)$ with $\underline{c}$ given in (3.24) *below is a solution, while for $a \geq 2$ the function $u \equiv 0$ is a solution.*

Proof: With the same reasoning as in the proof of Theorem 3.4 we may conclude that without loss of generality a solution satisfies the eiconal equation (3.21) under homogeneous boundary conditions, so that $|\nabla u| = const.$ in Ω. Now a one-dimensional optimization shows that for large α the surface area term is so dominant that it forces $|\nabla u|$ to vanish everywhere. In fact, for $c \geq 0$ set

$$g(c) := \frac{1}{1+c^2} + \alpha\ \sqrt{1+c^2}$$

and note that $g(0) = 1+\alpha$, $g'(0) = 0$ and $g(\infty) = \infty$. Moreover, for $\alpha < 2$ the function g has a global minimum at

$$\underline{c} = \sqrt{\left(\frac{2}{\alpha}\right)^{2/3} - 1} \tag{3.24}$$

and

$$g(\underline{c}) = \left[\left(\frac{1}{2}\right)^{2/3} + 2^{1/3}\right]\alpha^{2/3} < 1+\alpha,$$

while for $\alpha \geq 2$ the function g is monotone increasing on $\mathbb{R}^+$.

IV. Extremal eigenvalue problems

Eigenvalues of differential operators on bounded domains depend on the underlying domain Ω. A well-known result is the Faber-Krahn inequality, which states that *among all membranes of given area the circular one has the lowest fundamental frequency.* A more precise way of stating this result is the following.

Theorem 4.1: Faber-Krahn inequality
Let $\lambda_1(\Omega)$ be the first eigenvalue for

$$\begin{aligned} \Delta u + \lambda u &= 0 \qquad \text{in } \Omega \subset \mathbb{R}^N, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

then the problem of minimizing $\lambda_1(\Omega)$ among all open sets of prescribed volume (i.e. N-dimensional Lebesgue measure) has a solution.
Moreover, the domain which minimizes λ_1 is a ball, i.e. $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$.

Here as well as later Ω^* denotes the ball of the same volume as Ω, centered at the origin. Notice that Theorem 4.1 contains two statements, namely one about the existence of a minimizing domain and the other one about its shape. There are two conceptually different ways to prove Theorem 1.

a) The first proof relies on a constructive procedure, by establishing the Faber-Krahn inequality directly. Suppose Ω is any admissible domain and u_1 is a positive first eigenfunction on Ω. Then we can rearrange u_1 into its Schwarz-symmetrization u_1^* as follows. Each level set $\Omega_c := \{x \in \Omega \mid u(x) > c\}$ is transformed into Ω_c^*, a ball of the same volume as Ω_c, and u_1^* is defined as the unique radially decreasing function whose level sets are given by Ω_c^*. Now Cavalieri's principle implies $\int_\Omega |u_1|^2 dx = \int_{\Omega^*} |u_1^*|^2 dx$, and (this is the hard part, see [36]) $\int_\Omega |\nabla u_1|^2 dx \geq \int_{\Omega^*} |\nabla u_1^*|^2 dx$. But now the validity of the Faber-Krahn inequality is apparent from the variational characterization of λ_1 and from the following chain of inequalities.

$$\begin{aligned} \lambda_1(\Omega) &= \frac{\int_\Omega |\nabla u_1|^2 dx}{\int_\Omega |u_1|^2 dx} \\ &\geq \frac{\int_{\Omega^*} |\nabla u_1^*|^2 dx}{\int_{\Omega^*} |u_1^*|^2 dx} \geq \inf_{v \in W_0^{1,2}(\Omega^*)} \frac{\int_{\Omega^*} |\nabla v|^2 dx}{\int_{\Omega^*} |v|^2 dx} = \lambda_1(\Omega^*) \end{aligned} \tag{4.2}$$

b) The second "proof" uses a necessary condition for minimizers. If there exists a minimizing domain Ω, then Ω must be a ball. In fact, if there exists a minimizing smooth domain Ω, then we can perturb its boundary in the direction of a vector field $V(x)$ in such a way, that the perturbation is volume preserving. Therefore, using $\nu(x)$ as normal vector to $\partial\Omega$,

$$\int_{\partial\Omega} (V(x) \cdot \nu(x)) ds = 0. \tag{4.3}$$

The first variation of λ_1 with respect to domain variations is well-known as Hadamard's formula, and since it vanishes, we have

$$\int_{\partial\Omega} |\nabla u_1|^2 (V(x) \cdot \nu(x)) ds = 0. \tag{4.4}$$

Derivations of the Hadamard formula are given in [15, p.262] or [58, p.138]. To apply [15], set $F(u, \nabla u) = |\nabla u|^2 - \lambda_1(\Omega)\, u^2$.

By the way, according to [28] Hadamard's condition was already known to Lord Rayleigh, who had conjectured the Faber-Krahn inequality more than a century ago in 1877. Relation (4.4) tells us that $|\nabla u_1|^2$ is orthogonal in $L^2(\partial\Omega)$ to the function $(V(x)\cdot\nu(x))$, but according to (4.3), $(V(x)\cdot\nu(x))$ has as its range the orthogonal complement to the constant function 1. Therefore (4.3) and (4.4) imply

$$\frac{\partial u_1}{\partial \nu} = const. \qquad \text{on } \partial\Omega \tag{4.5}$$

as additional boundary condition to (4.1). But due to a famous result of Serrin [56] the overdetermined boundary value problem (4.1) (4.5) can only have a solution if Ω is a ball.

Notice that approach b) contains two gaps. First of all it uses the assumption that there exists a minimizing domain Ω. Secondly it assumes smoothness of its boundary in the derivation of Hadamard's formula. A similar state of the art has been reached in a fourth order problem, which models a vibrating clamped plate rather than a membrane. Incidentally, both Theorems 4.1 and 4.2 had been conjectured by Lord Rayleigh in 1877, see [49, 48].

Theorem 4.2: Nadirashvili-Ashbaugh-Benguria inequality
Let $\gamma_1(\Omega)$ be the first eigenvalue in

$$\begin{aligned} \Delta\Delta u - \gamma u &= 0 \qquad in\ \Omega\subset\mathbb{R}^2, \\ u = \frac{\partial u}{\partial n} &= 0 \qquad on\ \partial\Omega, \end{aligned} \tag{4.6}$$

then the problem of minimizing $\gamma_1(\Omega)$ among all open sets of prescribed area (i.e. 2-dimensional Lebesgue measure) has a solution.
Moreover, the domain which minimizes γ_1 is a circular disc, i.e. $\gamma_1(\Omega)\geq\gamma_1(\Omega^*)$.

Again there are two conceptually different ways to prove the Theorem.

a) The first proof relies on a delicate rearrangement argument, rearranging functions like the restriction of Δu to the set on which u is positive and so on. After all, $\gamma_1(\Omega)$ is characterised via a Rayleigh quotient, namely

$$\gamma_1(\Omega) = \inf_{v\in W_0^{2,2}(\Omega)} \frac{\int_\Omega |\Delta v|^2 dx}{\int_\Omega |v|^2 dx}.$$

The foundation for this proof was laid by Talenti in [64], who had already shown that $\gamma_1(\Omega)\geq 0.97768\ \gamma_1(\Omega^*)$. Nadirashvili was able to improve his result for $N=2$ and thus finished a whole series of increasingly successful attempts to prove it completely. In fact Talenti, as well as later Ashbaugh and Laugesen, had attacked this problem for any space dimension N. For $N=3$ the result of Nadirashvili could be recovered by Ashbaugh and Benguria in [3], but for general $N\geq 4$ nowadays one has only an estimate of the type

$$\gamma_1(\Omega) \geq c_N \gamma_1(\Omega^*)$$

with $c_N < 1$.

b) The second "proof" uses a necessary condition for minimizers. If there exists a minimizing domain Ω then Ω must be a ball. This strategy had been chosen by Mohr prior to Talenti and was also suggested by me in [38, p.172], while I was still unaware of [46]. In fact, using a variant of the Hadamard formula Mohr arrives at

$$\int_{\partial\Omega} |\Delta u_1|^2 (V(x) \cdot \nu(x)) ds = 0 \tag{4.7}$$

instead of (4.4), and thus – using (4.6) – at

$$\Delta u_1 = \frac{\partial^2 u_1}{\partial \nu^2} = const. \qquad \text{on } \partial\Omega \tag{4.8}$$

as an additional boundary condition. He then proceeds to show that the overdetermined problem (4.6) (4.8) has a solution only if Ω is a circular disc.

As mentioned above, approach b) has two shortcomings, because it is based on the assumptions that there exists a minimizing domain Ω and that its boundary is smooth. There is a third eigenvalue inequality which has been considered in [55, Note F] and [63]. It is meant to describe the buckling of a clamped plate. For one space dimension this problem is known as buckling of the Euler rod. The eigenvalue problem is

$$\begin{aligned} \Delta\Delta u + \Lambda \Delta u &= 0 \qquad \text{in } \Omega \subset \mathbb{R}^2, \\ u = \frac{\partial u}{\partial n} &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{4.9}$$

and the variational characterization of its first eigenvalue Λ_1 reads

$$\Lambda_1(\Omega) = \inf_{v \in W_0^{2,2}(\Omega)} \frac{\int_\Omega |\Delta v|^2 dx}{\int_\Omega |\nabla v|^2 dx}.$$

After Theorems 4.1 and 4.2 it is only natural to ask if $\Lambda_1(\Omega) \geq \Lambda_1(\Omega^*)$. However, only partial answers to this question are curently available. The first one was given by Szegö himself, using Schwarz symmetrization.

Proposition 4.3: [63, 55]
If Ω is such that the first eigenfunction of (4.9) *is positive, then $\Lambda_1(\Omega) \geq \Lambda_1(\Omega^*)$.*

Unfortunately the assumption of Proposition 4.3 is known to be violated for many domains, for instance domains with corners, see [67]. Incidentally, Szegö could also prove Theorem 4.2 under this assumption, but there it is known to be violated again [67]. If one wants a lower bound for $\Lambda_1(\Omega)$ which does not depend on this positivity assumption, the best estimate up to now has been obtained by Ashbaugh and Laugesen in [5], who showed

$$\Lambda_1(\Omega) \geq c_N \ \Lambda_1(\Omega^*)$$

for general domains in $\mathbb{R}^N$, with special constants $\tilde{c}_N < 1$ and $\lim_{N\to\infty} \tilde{c}_N = 1$. In particular $\tilde{c}_2 \geq 0.7877$.

While Proposition 4.3 follows strategy a) above, there is also a (hitherto unpublished) approach of Willms following strategy b). N.B. Willms can be reached at Bishops University in Quebec and he has kindly allowed me to reproduce his proof here. The final chain of inequalities in this proof was strung together by B.Willms and H.Weinberger, and H.Weinberger as well agreed to its publication here.

Proposition 4.4: [68]
If there exists a domain $\tilde{\Omega}$ which minimizes Λ_1 among all plane domains of prescribed area and which has a smooth boundary $\partial\tilde{\Omega}$, then $\tilde{\Omega}$ must be a circular disc, in other words then $\Lambda_1(\Omega) \geq \Lambda_1(\Omega^*)$ holds for any admissible Ω.

The proof of Proposition 4.4 is tricky and was presented by Willms at an Oberwolfach meeting in 1995. Suppose Ω minimizes Λ_1 among all plane domains of given area and $\partial\Omega$ is smooth. Again we denote the associated eigenfunction by u_1. Then from a domain variation as in [48] or [15], one can conclude (4.7) or

$$\Delta u_1 = c \quad \text{on } \partial\Omega, \tag{4.10}$$

where c is a real constant. Now we want to show that the overdetermined boundary value problem (4.9) (4.10) can only have a solution if Ω is a circular disc.

To do this we use the substitution $z := \Delta u_1 + \Lambda_1 u_1$. Then the differential equation from (4.9) and the boundary condition (4.10) become

$$\Delta z = 0 \quad \text{in } \Omega, \qquad z = c \quad \text{on } \partial\Omega,$$

so that $z \equiv c$ by the maximum principle for harmonic functions. Next we use the substitution

$$v(x) = u_1(x) - \frac{c}{\Lambda_1},$$

so that (4.9) and (4.10) are transformed into the overdetermined boundary value problem

$$\left.\begin{aligned} \Delta v + \Lambda_1 v &= 0 && \text{in } \Omega, \\ v &= -\frac{c}{\Lambda_1} && \text{on } \partial\Omega, \\ \frac{\partial v}{\partial \nu} &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{4.11}$$

Problem (4.11) is reminiscent of the (unsolved) Pompeiu problem (4.12),

$$\left.\begin{aligned} \Delta v + \nu v &= 0 && \text{in } \Omega, \\ v &= c && \text{on } \partial\Omega, \\ \frac{\partial v}{\partial \nu} &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{4.12}$$

in which one wants to show that an eigenfunction v of the Laplace operator under Neumann boundary conditions, which in addition is constant on the boundary, can only exist if the underlying domain is a ball. For a recent partial result on this problem I refer to [29]. It was also shown in [6] that for problem (4.12) the domain Ω must be a ball, provided $\nu < \lambda_2$, the second Dirichlet eigenvalue of the Laplace operator from (4.1). The eigenvalue in (4.11) violates this constraint because an inequality due to Payne [52, 53, Thm. 6] states that

$$\Lambda_1(\Omega) \geq \lambda_2(\Omega), \tag{4.13}$$

and equality holds only if Ω is a ball, in which case there is nothing to prove. Nevertheless we will conclude from (4.11) that Ω must be a circular disc after all.

To this end let us first observe that by translation of Ω we can make sure that the origin is contained in Ω, and that v has a critical point at the origin. Then we follow an idea that has been used by Payne [52] and Aviles [6] and set $w := xv_y - yv_x$. In polar coordinates (r, θ) this can be read as $w = rv_\theta$. Now a simple calculation shows

$$\begin{aligned} \Delta w + \Lambda_1 w &= 0 && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.14}$$

Therefore either $w \equiv 0$ or $\Lambda_1 = \lambda_k(\Omega)$, a Dirichlet eigenvalue for the Laplace operator on Ω and $k \geq 3$. In the first of these cases $v_\theta \equiv 0$, i.e. v depends only on r and thus Ω is a circular disc. Thus the proof of Proposition 4.4 is complete, provided we can rule out the second case. In the second case we note that both w and ∇w vanish in the origin by construction. In fact $w_x = v_y + xv_{xy} - yv_{xx}$, and $v_y(0) = 0$, so $w_x(0) = 0$, and similarly $w_y = 0$. This means that the origin is a nodal point of w, and a point where a nodal line intersects itself transversally. But then for topological reasons this nodal line divides Ω into at least three nodal domains, at least one of which has area not exceeding $|\Omega|/3$. But now we get the following chain of inequalities

$$\begin{aligned} \Lambda_1(\Omega) = \lambda_k(\Omega) &= \lambda_1(\text{subdomain of area } \leq |\Omega|/3) \\ &\geq \lambda_1(\text{subdomain of area } = |\Omega|/3) \\ &\geq \lambda_1(\text{disc of area } = |\Omega|/3) \\ &= 3\lambda_1(\text{disc of area } = |\Omega|) \\ &> \lambda_2\ (\text{disc of area } = |\Omega|) \\ &= \Lambda_1(\text{disc of area } = |\Omega|), \end{aligned}$$

which contradicts the minimality of Λ_1 in Ω. In this chain of inequalities we have used one after the other the monotonicity of λ_1 with respect to the domain, the Faber-Krahn inequality, a scaling argument, the Payne-Polya-Weinberger inequality from [54], see also [2,4], and the discussion of the equality sign in (4.13).

So much for the proof of Proposition 4.4. The discomforting aspect of approach b), that existence of a minimizing domain is assumed, can be partly resolved in several ways. There is a general existence result by Buttazzo and Dal Maso, according to which a minimizing domain of prescribed area exists among all quasi-open sets, see [13]. A more satisfactory answer might be provided by the following Proposition, whose more geometric proof is taken from [16, Sect. 2], where it was stated for the special case $\alpha = 1$.

Proposition 4.5: [16]
Among convex plane domains of prescribed area, the eigenvalues $\lambda_1(\Omega)$ from (4.1), $\gamma_1(\Omega)$ from (4.6) and $\Gamma_1(\Omega)$ from (4.9) have a minimizer. In fact any functional $j(\Omega)$ which has the property that it is
i) bounded below,
ii) monotone in the sense $j(\omega) > j(\Omega)$ for $\omega \subset \Omega$, and
iii) coercive in the sense that it tends to ∞ on strips whose width tends to 0,
iv) homogeneous of degree -2α in the sense that there exists an $\alpha > 0$ such that $j(t\Omega) = t^{-2\alpha} j(\Omega)$ for every $t > 0$,
can be minimized among convex plane domains Ω of prescribed area.

For the proof we set $A := \{\ G \subset \mathbb{R}^2 \ |G \text{ convex},\ |G| = c > 0\ \}$ and choose a sequence G_j from A which minimizes λ_1 (or γ_1 or Λ_1). Notice that for λ_1 and Λ_1 we have to pick $\alpha = 1$, while for γ_1 one has to choose $\alpha = 2$ in iv). We shall only use the properties of a general j.
The sequence G_j has width $w(G_j)$ uniformly bounded below by a positive constant,

$$w(G_j) \geq \delta > 0, \tag{4.15}$$

because otherwise G_j fits into arbitrarily thin strips S_j of width ε_j with $\varepsilon_j \to 0$, but due to properties i) and ii) this would imply $j(G_j) \geq j(S_j) \to \infty$, which contradicts the property that the sequence G_j is a minimizing sequence. The width of a convex domain is the minimal distance of two parallel supporting hyperplanes.
After possible translation, the sequence G_j, is uniformly bounded, i.e. without loss of generality we may assume that there exists an $R < \infty$ such that

$$G_j \subset B_R(0) \tag{4.16}$$

for all G_j. Otherwise, since each G_j is convex and of prescribed area, it would become too thin, which contradicts (4.15).

According to a result of Sholander [59, Thm. 20], for any plane convex set $6\ |G| \geq |\partial G|\ w(G)$. This and (4.15) imply that the perimeters $|\partial G_j|$ of G_j are uniformly bounded above.
Therefore Bonnesen's inequality, see e.g. [Ba, p.8], implies that a ball of uniform radius $\rho > 0$ fits into each G_j, because $\pi\rho^2(G_j) + |G_j| \leq |\partial G_j|\rho(G_j)$ holds for the radius $\rho(G_j)$ of the largest disc in G_j. Thus the boundary of G_j can be represented in polar coordinates by a 2π-periodic function $r_j(\theta)$, and $r_j(\theta) \in [\rho, R]$ for any natural j and any $\theta \in [0, 2\pi]$.
The fact that $B_\rho(0) \subset G_j \subset B_R(0)$ and the convexity of G_j imply that G_j is starshaped with respect to every $x \in B_\rho(0)$ and that the sequence r_j is equi-Lipschitz continuous in θ. Hence by the Arzela-Ascoli theorem $r_j \to r_\infty$ uniformly, $G_j \to G_\infty$ in a strong topology, and $2|G_\infty| = \int_0^{2\pi}[r_\infty(\theta)]^2 d\theta = c$, so that G_∞ is admissible.
Finally $j(G_\infty) \leq \liminf j(G_j)$ by a continuity of j with respect to this convergence, which was shown in [17, Lemma 2.1] for the case $\alpha = 1$. In fact, if $||r_k - r_m||_\infty = d \leq \rho$ then $r_k = d + (r_k - d) \leq r_j + d = r_j(1 + d/\rho)$ so that

$$\begin{aligned} j(\Omega_m) &\leq j(\Omega_k(1 + d/\rho)^{-1}) \\ &= (1 + d/\rho)^{2\alpha} j(\Omega_k) \\ &\leq (1 + c_\alpha\ d/\rho) j(\Omega_k) \\ &\leq j(\Omega_k) + (c_\alpha/\rho) j(B_\rho)\ d \\ &= j(\Omega_k) + (c_\alpha/\rho) j(B_\rho) ||r_k - r_m||_\infty. \end{aligned} \tag{4.17}$$

In this chain of inequalities we have one after the other used the domain monotonicity i), the homogeneity iv), the fact that on the interval $(0, 1)$ the function $(1 + s)^{2\alpha}$ can be bounded by a linear function $1 + c_\alpha s$, and again the monotonicity property i). Reversing the role of Ω_m and Ω_k gives

$$|j(\Omega_m) - j(\Omega_k)| \leq (c_\alpha/\rho) j(B_\rho) ||r_k - r_m||_\infty, \tag{4.18}$$

that is the strong continuity of j with respect to this domain convergence.

While Proposition 4.5 provides us with a convex, and thus a minimizing Lipschitz domain, I cannot prove more regularity of this minimizer. Is Ω strictly convex? Is its boundary smooth enough to derive the Hadamard condition? These appear to be open questions.

Acknowledgement: The results presented in these notes are to a great part due to others, among them notably N.B.Willms and H.Weinberger (Proposition 4.4) and my coauthors M.Belloni, G.Buttazzo, F.Brock and V.Ferone. It is a pleasure to thank various colleagues for helpful discussions that led to an improvement of these notes, namely M.Ashbaugh (Columbia, MO) and M.Belloni (Parma). Finally I thank my colleagues S.Fekete and K.Veselic for providing me with copies of [1, 19] and [32] and my students M.Mester, N.Oll and A.D.Stolz for their calculations.

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