Variations on the p-Laplacian

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This paper is dedicated to Jean Pierre Gossez on the occasion of his 65th birthday.

ABSTRACT. In this paper I address several issues involving Dirichlet problems for the classical p–Laplacian operator $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ for $p \in (1,\infty)$. First I look at p harmonic functions as $p \to \infty$ and $p \to 1$. Then I compare the p–Laplacian with its normalized version $\Delta_p^N u := \frac{1}{p}|\nabla u|^{2-p}\Delta_p u$ and study equations like $-\Delta_p u = 1$ or $-\Delta_p^N u = 1$. Finally I present results and open problems on the eigenvalue problem $-\Delta_p u = \lambda|u|^{p-2}u$.

1. Preliminaries on intrinsic coordinates

Let me start with the observation, that for a function u with nonvanishing gradient one can write the Laplacian of u as follows

$$\Delta u = u_{x_1x_1} + \ldots + u_{x_nx_n} = u_{\nu\nu} + u_\nu \ \mathrm{div}(\nu) \tag{1.1}$$

where $\nu(x) = -\frac{\nabla u(x)}{|\nabla u(x)|}$ is the direction of steepest descent. In fact,

$$\mathrm{div}(\nu) = -\frac{\Delta u}{|\nabla u|} + \frac{u_{x_i}u_{x_j}u_{x_ix_j}}{|\nabla u|^3} = -\frac{\Delta u}{|\nabla u|} + \frac{u_{\nu\nu}}{|\nabla u|}$$

so that $\Delta u = u_{\nu\nu} - |\nabla u| \ \mathrm{div}(\nu) = u_{\nu\nu} + u_\nu \ \mathrm{div}(\nu)$ or

$$\Delta u = u_{\nu\nu} + u_\nu \ (n-1)H \tag{1.2}$$

with H denoting mean curvature of a level set of u. To avoid misunderstandings, here the sign of $H(\{x; u(x) = t\})$ is nonnegative if $\{x; u(x) \geq t\}$ is convex. For radially decreasing functions u one recovers the well known representation of the Laplacian in polar coordinates $\Delta u = u_{rr} + \frac{n-1}{r}u_r$.

In a similar fashion, for $p \in (1,\infty)$ one can write the p-Laplacian of u as

$$\begin{aligned} \Delta_p u &= \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = |\nabla u|^{p-2} \ [\Delta u + (p-2)u_{\nu\nu}] && (1.3)\\ &= |\nabla u|^{p-2} \ \ [(p-1)u_{\nu\nu} + (n-1)Hu_\nu] && (1.4)\end{aligned}$$

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and the *normalized* or *game-theoretic p-Laplacian* as

$$\Delta_p^N u = \tfrac{1}{p}|\nabla u|^{2-p}\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = \tfrac{p-1}{p}u_{\nu\nu} + \tfrac{1}{p}(n-1)Hu_\nu \tag{1.5}$$

$$= \frac{p-1}{p}\Delta_\infty^N u + \frac{1}{p}\Delta_1^N u\ . \tag{1.6}$$

Observe that $\Delta_\infty^N u = u_{\nu\nu}$, while $\Delta_2^N u = \frac{1}{2}\Delta u$ and $\Delta_1^N u = |\nabla u|\mathrm{div}(\frac{\nabla u}{|\nabla u|})$. Therefore $\Delta_p^N u$ turns out to be a convex combination of $\Delta_\infty^N u$ and $\Delta_1^N u$. One purpose of this manuscript is the comparison of the normalised with the classical p-Laplacian operator.

2. The Dirichlet problem for p–harmonic functions

Suppose that $\Omega \subset \mathbb{R}^n$ is bounded and connected, $\partial\Omega$ of class $C^{2,\alpha}$ and $g(x) \in W^{2,p}(\Omega)$. Consider the Dirichlet problem

$$-\Delta_p u = 0 \qquad \text{in } \Omega, \tag{2.1}$$

$$u(x) = g(x) \qquad \text{on } \partial\Omega. \tag{2.2}$$

This problem is well understood for $p \in (1,\infty)$. In fact, u can be charactzerized as the unique (weak) solution of the strictly convex variational problem

$$\text{Minimize } \ I_p(v) = ||\nabla v||_{L^p(\Omega)} \quad \text{on } g(x) + W_0^{1,p}(\Omega), \tag{2.3}$$

so that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi\ dx = 0 \text{ for every } \phi \in W_0^{1,p}(\Omega). \tag{2.4}$$

It is well known, that weak solutions are locally of class $C^{1,\alpha}$. They are even of class C^∞ wherever their gradient does not vanish, see e.g. [**L**] and references therein.

One can show [**JLM**] that weak solutions are also viscosity solutions of the associated Euler equation

$$F_p(Du, D^2u) = -|Du|^{p-4}\left(|Du|^2\mathrm{trace}D^2u + \langle D^2uDu, Du\rangle\right) = 0. \tag{2.5}$$

Definition 2.1. Following [**CIL**], $u \in C(\Omega)$ is a **viscosity solution** of the equation $F(Du, D^2u) = 0$, if it is both a viscosity subsolution and a viscosity supersolution.

u is a **viscosity subsolution** of $F(Du, D^2u) = 0$, if for every $x \in \Omega$ and $\varphi \in C^2$ such that $\varphi - u$ has a minimum at x, the inequality $F_*(D\varphi, D^2\varphi) \leq 0$ holds. Here F_* is the lower semicontinuous hull of F.

u is a **viscosity supersolution** of $F(Du, D^2u) = 0$, if for every $x \in \Omega$ and $\varphi \in C^2$ such that $\varphi - u$ has a maximum at x, the inequality $F^*(D\varphi, D^2\varphi) \geq 0$ holds. Here F^* is the upper semicontinuous hull of F.

Incidentally, only for $p \in (1,2)$ does this imply that they are also viscosity solutions of the normalized equation

$$F_p^N(Du, D^2u) = -\frac{1}{p}\mathrm{trace}D^2u - \frac{p-2}{p}\,\frac{\langle D^2uDu, Du\rangle}{|Du|^2} = 0. \tag{2.6}$$

What happens as $p \to \infty$? For $g \in W^{1,\infty}(\Omega)$ the $W^{1,p}$-norm of the family $\{u_p\}$ is uniformly bounded, because $I_p(u_p) \leq I_p(g) \leq ||\nabla g||_\infty|\Omega|$. Moreover, for $q > n$ fixed and $p > q$ one finds

$$||\nabla u_p||_q \leq ||\nabla u_p||_p\ |\Omega|^{(p-q)/pq} \leq ||\nabla g||_\infty|\Omega|^{1+1/q},$$

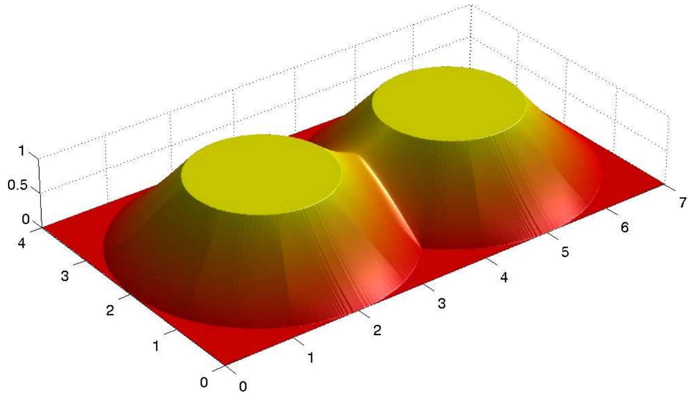

FIGURE 1. Graph of the function $u_\infty(x)$.

a bound in $W^{1,q}(\Omega)$ independent of p, so that by the Sobolev embedding theorem $u_p \to u_\infty$ in some C^α.

According to the stability theorem for viscosity solutions u_∞ should be viscosity solution to a limit equation $F_\infty(Du, D^2u) = 0$. What is this equation? Let us check the condition for subsolutions.

Let φ be a C^2 testfunction s.th. $\varphi - u_\infty$ has a min at x_∞ and $\nabla\varphi(x_\infty) \neq 0$. Then without loss of generality $\varphi - u_p$ has a min at x_p near x_∞ and $x_p \to x_\infty$ as $p \to \infty$. Since u_p is a viscosity subsolution of $F_p(Du, D^2u) = 0$, we have

$$-|D\varphi|^{p-4}\left[|D\varphi|^2\Delta\varphi + (p-2)\langle D^2\varphi D\varphi, D\varphi\rangle\right](x_p) \leq 0,$$

or

$$-\frac{p-2}{p}\langle D^2\varphi D\varphi, D\varphi\rangle(x_p) \leq \frac{1}{p}|D\varphi|^2\Delta\varphi(x_p).$$

Observing that $\varphi \in C^2$ and sending $p \to \infty$ gives the desired inequality

$$-\langle D^2\varphi D\varphi, D\varphi\rangle(x_\infty) := -\Delta_\infty\varphi \leq 0$$

for viscosity subsolutions of $F_\infty(Du, D^2u) = 0$. The proof for supersolutions is analogous. Thus u_∞ is (unique) viscosity solution of $-\Delta_\infty u = 0$ in Ω. Uniqueness follows from a celebrated result of Jensen [**Je**], for which there are various proofs available now [**BB, ACJ, CGW, AS**].

It is worth noting that the variational problem

$$\text{Minimize} \quad I_\infty(v) = ||\nabla v||_{L^\infty(\Omega)} \quad \text{on } g(x) + W_0^{1,\infty}(\Omega), \tag{2.7}$$

can have **many** solutions. To give an example, let Ω consist of two overlapping discs of radius 2 minus two smaller discs of radius 1, concentrical to the large discs. Then Ω has the shape of a figure 8. Suppose that $g(x) = 0$ on the outer part of this boundary, $g(x) = 0$ on the two small circles inside, and $g \in W^{\cdot}\infty(\Omega)$. Then Problem (2.7) can have the minimum of two cones as a solution (which is not of class C^1) or the infinite-harmonic function u_∞ with boundary values g that is depicted in Figure 1. But by results of Savin and Evans/Savin [**Sa, ES**] u_∞ is of class C^1 and even $C^{1,\alpha}$.

What happens to p-harmonic functions as $p \to 1$? In general, one cannot expect uniform convergence of u_p, but Juutinen [**Ju**] found sufficient conditions for it.

THEOREM 2.2. *If $g \in C(\overline{\Omega})$ and Ω convex, then $u_p \to u_1$ uniformly as $p \to 1$. Moreover, u_1 is unique minimizer of*

$$E_1(v) = \sup\left\{ \int_\Omega u \ \mathrm{div}\sigma dx;\ \sigma \in C_0^\infty(\Omega, \mathbb{R}^n),\ |\sigma(x)| \le 1 \ in\ \Omega \right\}$$

on $\{v \in BV(\Omega) \cap C(\overline{\Omega}),\ v = g \ on\ \partial\Omega\}$.

This time the limiting variational problem

$$\text{Minimize } I_1(v) = \int_\Omega |Dv|\ dx + \int_{\partial\Omega} |v - g(x)|\ d\sigma \quad \text{on } BV(\Omega), \tag{2.8}$$

has a **unique** solution, while the limiting Euler equation can have **many** viscosity solutions.

Let us first give a heuristic reason for the **uniqueness** of any minimizer of the TV-functional. If there are two minimizers u and v (for simplicity in $W^{1,1}(\Omega)$) of I_1, then any convex combination $w = tu + (1-t)v$ would also be minimizer, hence level lines of u are also level lines of v, and ∇u is always parallel to ∇v. Consequently $v = f(u)$. Now the Dirichlet condition $v = g = u$ on $\partial\Omega$ implies $f(g) = g$, so that $f = Id$ on $range_{\partial\Omega}(g)$. But since both u and v are bounded below (and above) in Ω by $\min_{\partial\Omega} g$ (and $\max_{\partial\Omega} g$) in Ω we find $f(u) = u$. To prove that minimizers of 2.8 satisfy the maximum principle one cuts them off by $\max_{\partial\Omega} g$. This would decrease the functional if the maximum principle were violated.

Now I should explain why there can be **nonuniqueness** of viscosity solutions to the Dirichlet problem

$$-\Delta_1 u = 0 \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega. \tag{2.9}$$

Sternberg & Ziemer gave the following counterexample in [**SZ**]: Let $\Omega = B(0,1) \in \mathbb{R}^2$ be the unit disc in the plane and suppose that on $\partial\Omega$ the function g is given in polar coordinates by $g(x_1, x_2) = \cos(2\varphi)$.

Then (2.9) has a whole family u_λ of viscosity solutions, $\lambda \in [-1, 1]$, but only one of them, u_0 minimizes I_1. In fact, the function

$$u_\lambda(x_1, x_2) := \begin{cases} 2x_1^2 - 1 & \text{left and right of rectangle in Figure 2,} \\ \lambda & \text{in the rectangle generated by } \cos(2\varphi) = \lambda, \\ 1 - 2x_2^2 & \text{on top and bottom of this rectangle,} \end{cases}$$

is a viscosity sol. of both $-\Delta_1 u = 0$ and $-\Delta_1^N u = \kappa|\nabla u| = 0$ in Ω.

Verifying this is a delicate matter, and I will do so for the equation $-\Delta_1^N u = 0$. Notice that the functon F_p^N from (2.6) is discontinuous at $Du = 0$, so that one has to refine the notion of viscosity solutions of $F(Du, D^2u)$ for discontinuous F as prescribed in Definition 2.1. Now since F_p^N is given by

$$F_p^N(q, X) = \begin{cases} -\frac{1}{p}\left(\delta_{ij} + (p-1)\frac{q_i q_j}{|q|^2}\right) X_{ij} & \text{if } q \neq 0 \\ ? & \text{if } q = 0 \end{cases}$$

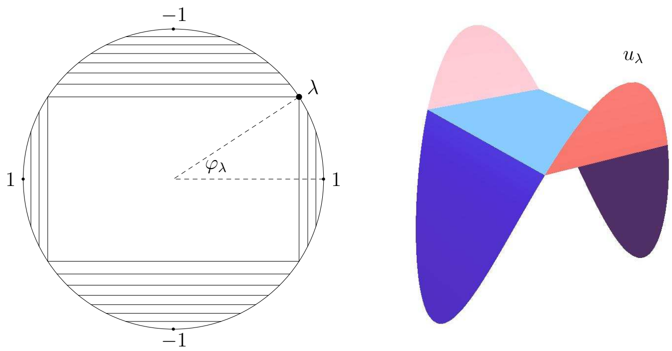

FIGURE 2. Level lines and graph of the function $u_\lambda(x)$.

we have to compute its semicontinuous limits as $q \to 0$. Each symmetric matrix X has real eigenvalues, and we order them according to magnitude as $\lambda_1(X) \leq \lambda_2(X) \leq \ldots \leq \lambda_n(X)$. Then a simple calculation shows that

$$F^N_{p\ *}(0,X) = \begin{cases} -\frac{1}{p}\sum_{i=1}^{n-1}\lambda_i - \frac{p-1}{p}\lambda_n & \text{if } p \in [2,\infty] \\ -\frac{1}{p}\sum_{i=2}^{n}\lambda_i - \frac{p-1}{p}\lambda_1 & \text{if } p \in [1,2] \end{cases} \tag{2.10}$$

$$F^{N*}_{p}(0,X) = \begin{cases} -\frac{1}{p}\sum_{i=2}^{n}\lambda_i - \frac{p-1}{p}\lambda_1 & \text{if } p \in [2,\infty] \\ -\frac{1}{p}\sum_{i=1}^{n-1}\lambda_i - \frac{p-1}{p}\lambda_n & \text{if } p \in [1,2] \end{cases} \tag{2.11}$$

In particular, for $n = 2$ and $p = 1$ we get $F^N_{1\ *}(0,X) = -\lambda_2$, so that we require $-\lambda_2(D^2\varphi) \leq 0$ for subsolutions, whenever $\nabla\varphi(x) = 0$. In fact, when φ touches the graph of u_λ from above (at the front left edge in Figure 2), then at least one of the principal curvatures if the graph of φ must be nonnegative. In particular $\lambda_2(D^2\varphi)$ is nonnegative as requested for subsolutions. By dual reasoning $F^{N*}_1(0,X) = -\lambda_1$, and $-\lambda_1(D^2\varphi) \geq 0$ for supersolutions if $\nabla\varphi(x) = 0$. Again one can see this, because even test functions φ touching u_λ from below at the front right edge in Figure 2), must have a graph with at least one of its principal curvatures and a fortiori $\lambda_1(D^2\varphi)$ nonpositive at the point of tangency.

3. The Dirichlet problem for $-\Delta_p u = 1$

The Dirichlet problem

$$-\Delta_p u_p = 1 \quad \text{in } \Omega, \tag{3.1}$$

$$u_p = 0 \quad \text{on } \partial\Omega \tag{3.2}$$

can be treated in a similar way. Again there are surprises as $p \to \infty$ or $p \to 1$. In fact it was shown in [**K1**] that

$$\lim_{p\to\infty} u_p(x) = d(x, \partial\Omega) \tag{3.3}$$

and in [**BDM**] that the limiting differential equation (in the sense of viscosity solutions) and boundary value problem is

$$|Du| = 1 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3.4}$$

It is well-known that the eikonal problem (3.4) has a unique viscosity solution, but many distributional solutions. The behaviour of u_p as $p \to 1$ is even more enigmatic.

$$\lim_{p\to 1} u_p(x) = \begin{cases} 0 & \text{if } \Omega \text{ is small} \\ \textit{discontinuous} & \text{if } \Omega \text{ is inbetween} \\ +\infty & \text{if } \Omega \text{ is large} \end{cases}$$

Why is there this strange behaviour as $p \to 1$? The answer lies in the variational formulation of (3.1). For $p > 1$ the solution of (3.1) is the unique minimizer of

$$E_p(v) := \int_\Omega \frac{1}{p}|\nabla v|^p - v\ dx \qquad \text{on } W_0^{1,p}(\Omega). \tag{3.5}$$

As $p \to 1$, this functional Γ-converges to

$$E_1(v) := \int_\Omega |Dv| - v\ dx + \int_{\partial\Omega} v\ d\sigma \qquad \text{on } BV(\Omega), \tag{3.6}$$

and by the coarea formula and Cavalieri's principle

$$E_1(v) := \int_{\mathbb{R}} [|\partial\{x \in \Omega;\ v(x) > t\}| - |\{x \in \Omega;\ v(x) > t\}|]\ dt. \tag{3.7}$$

Here we have used the notation that $|\partial D|$ denotes the perimeter of $D(\subset \Omega)$ in $\mathbb{R}^N$ and $|D|$ the volume or n-dimensional Lebesgue measure of D. Now the volume of a set scales with power n, while the perimeter scales with power $(n-1)$. Therefore, when Ω is large, the integrand in (3.7) can become negative, while for small Ω it must be positive unless the set $\{x \in \Omega;\ v(x) > t\}$ is a nullset for almost every t. So for small Ω the only minimizer of E_1 is the nullfunction, while for large Ω the functional E_1 is unbounded from below. For some intermediate size of Ω, however, the positive and negative part in E_1 are equal and then there exists a positive cone of nontrivial minimizers, spanned by a characteristic function of a special subset of Ω.

Definition 3.1. A set C_Ω is a Cheeger set of Ω if it infimizes the ratio $|\partial D|/|D|$ among all smooth subsets of Ω. Here $|\partial D|$ denotes the perimeter of D in $\mathbb{R}^N$ and $|D|$ the volume or n-dimensional Lebesgue measure of D.

Remark 3.2. Formally, the limiting equation $-\Delta_1 u = 1$ reads $(n-1)H = 1$ or $H = \frac{1}{n-1}$ in intrinsic coordinates. Level surfaces satisfying this curvature condition in Ω are boundaries of Cheeger sets. For some domains Ω such as balls the Cheeger set C_Ω coincides with Ω, but for other domains such as rectangles they are genuine subsets. In that case, and for Ω suitably scaled, $u_p(x) \to \chi_{C_\Omega}(x)$ as $p \to 1$, and then we have a discontinuous viscosity solution of $-\Delta_1 u = 1$, whose lower semicontinuous version is a viscosity supersolution, while the upper semicontinuous version is a viscosity subsolution. In fact, touching such a discontinuous function at an upper edge in Ω from above by a smooth testfunction φ, whose gradient vanishes at the point of contact, requires precisely $-\lambda_2(D^2\varphi) \leq 1$ or $\lambda_2(D^2\varphi) \geq -1$. Here we have used the notation from Section 2

Incidentally, Cheeger sets have many interesting properties, some of which are described in [**KF, KL, KN**].

In contrast to the equation $-\Delta_p u = 1$ relatively little is known about the Dirichlet problem

$$-\Delta_p^N u_p = 1 \quad \text{in } \Omega, \tag{3.8}$$

$$u_p = 0 \quad \text{on } \partial\Omega. \tag{3.9}$$

For $p \in (1, \infty]$ it was shown by Lu and Wang [**LW1, LW2**] that there exists a unique viscosity solution. It is tempting to think that the equations $-\Delta_p^N u = 1$ and $-\Delta_p u = p|\nabla u|^{p-2}$ are equivalent in the sense that the associated Dirichlet problems have the same solutions. This is not necessarily the case, because any constant function solves the latter equation. For $p = \infty$ equation (3.8) reads $-u_{\nu\nu} = 1$ in Ω and for $p = 1$ it turns into $|\nabla u|(n-1)H = 1$ in Ω. Both equations are degenerate elliptic in the sense of viscosity solutions.

4. Overdetermined Problems

Serrin and Weinberger proved in 1971 that the following overdetermined boundary value problem cannot have a solution in a smooth simply connected domain unless Ω is a ball.

$$\begin{cases} -\Delta u = 1 \quad \text{in } \Omega, \\ u = 0 \quad \text{and} \quad -\frac{\partial u}{\partial \nu} = a = const. \quad \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

If u denotes the velocity potential of laminar flow, then the result implies in two dimensions, that laminar flow in a noncircular pipe cannot have constant shear stress on the wall of the pipe.

Serrin's proof [**Se**] uses the moving plane method and applies to positive classical solutions of autonomous strongly elliptic equations

$$-\sum_{i,j=1}^{n} a_{ij}(u, |\nabla u|)u_{x_i x_j} = f(u, |\nabla u|),$$

while Weinberger's proof [**We**] is given only for $-\Delta u = 1$ and uses both variational methods and (other) maximum principles.

THEOREM 4.1. *Although the proof has to be modified, the result of Serrin and Weinberger applies also to the equation* $-\Delta_p u = 1$.

There have been several attempts to attack this problem, and the history of it as well as other generalizations are described in [**FK**]. I shall now outline the ideas of the proof, which contains essentially three steps.

1) The function $P(x) := \frac{2(p-1)}{p}|\nabla u(x)|^p + \frac{2}{n}u(x)$ attains its maximum over $\overline{\Omega}$ on $\partial\Omega$, and thus $P(x) \leq \frac{2(p-1)}{p}a^p =: c$ in Ω.
2) Show that $\int_\Omega P(x)dx = c|\Omega|$, then by Step 1) $P(x) \equiv c$ on $\overline{\Omega}$.
3) Show that $P \equiv c$ in Ω implies radial symmetry of u.

Steps 1) and 2) are not as straightforward as one might think. To prove Step 1) it is natural to strive for an inequality of type $-\Delta P + \ldots \leq 0$ in Ω. This is problematic, since in general $u \notin C^3$. A way out of this malaise is a suitable regularization of the problem by a class of regular elliptic equations, whose corresponding P_ε-functions satisfy the maximum principle. Then one can pass to the limit, see [**FK**]. To prove Step 2) one would like to use Pohožaev identities, but the classical versions of those need C^2-regularity of solutions, while our solutions are only $C^{1,\alpha}$. Fortunately Degiovanni, Musesti, Squassina were able to show in [**DMS**], that C^1 regularity suffices to perform the following chain of calculations, which provides a proof of Step 2, that $P \equiv c$ in Ω:

Testing $-\Delta_p u = 1$ with u gives

$$\int_\Omega |\nabla u|^p \; dx = \int_\Omega u \; dx, \tag{4.2}$$

while testing with $(x, \nabla u)$ gives

$$-\int_\Omega \Delta_p u(x,\nabla u) \; dx = \int_\Omega (x, \nabla u \; dx) = -n \int_\Omega u \; d\sigma \tag{4.3}$$

Under various integrations by part the left hand side of (4.3) is transformed as follows

$$\begin{aligned}
\text{lhs of (4.3)} &= \int_\Omega |\nabla u|^{p-2}\nabla u \nabla(x,\nabla u) \; dx - \int_{\partial\Omega} a^{p-2} u_\nu (x,\nabla u) \; d\sigma \\
&= \int_\Omega |\nabla u|^{p-2}\left[|\nabla u|^2 + (x, \nabla(\frac{|\nabla u|^2}{2}))\right] \; dx - \int_{\partial\Omega} a^p (x,\nu) \; d\sigma \\
&= \int_\Omega |\nabla u|^p + (x, \nabla(\frac{|\nabla u|^p}{p})) \; dx \; - \; a^p \; n \; |\Omega| \\
&= \int_\Omega |\nabla u|^p - n\frac{|\nabla u|^p}{p} \; dx + \int_{\partial\Omega} \frac{a^p}{p}(x,\nu) \; d\sigma \; - \; a^p \; n \; |\Omega| \\
&= \int_\Omega n\left[\frac{1}{n}|\nabla u|^p - \frac{|\nabla u|^p}{p}\right] dx - \; \frac{p-1}{p} a^p \; n \; |\Omega|
\end{aligned}$$

so that

$$\frac{2}{n} \; (4.3) \; = \int_\Omega \frac{2}{n}|\nabla u|^p - \frac{2}{p}|\nabla u|^p dx - c|\Omega| = -2\int_\Omega u.$$

Together with (4.2) we arrive at the identity

$$\int_\Omega \frac{2}{n}u + \frac{2(p-1)}{p}|\nabla u|^p \; dx = c|\Omega| \quad (= \int_\Omega P(x) \; dx)$$

which establishes Step 2. This and Step 1 imply that $P \equiv c$ in Ω. It remains to prove Step 3, that $P \equiv c$ in Ω implies symmetry. For this I distinguish two cases
a) If $\partial\Omega \in C^{2,\alpha}$, then$P_\nu = 0$ on $\partial\Omega$ implies $H = \frac{1}{n}a^{1-p}$, because the two identities

$$P_\nu = 2(p-1)|u_\nu|^{p-2}u_\nu u_{\nu\nu} + \frac{2}{n}u_\nu = \left[(p-1)|u_\nu|^{p-2}u_{\nu\nu} + \frac{1}{n}\right] 2u_\nu = 0$$

and

$$\Delta_p u = -1 = (p-1)|u_\nu|^{p-2}u_{\nu\nu} + (n-1)H|u_\nu|^{p-2}u_\nu$$

imply $H = \frac{1}{n}a^{1-p}$ on $\partial\Omega$. Hence $\partial\Omega$ has constant mean curvature and by a famous theorem of Alexandrov Ω must be a ball.
b) If $\partial\Omega$ is not smooth, consider $\Gamma := \{x \mid u(x) = \varepsilon\}$. Since $u \in C^{1,\beta}(\Omega)$ and $u_\nu = -a$ on $\partial\Omega$, we know that $\nabla u \neq 0$ and $u \in C^{2,\beta}$ near Γ. Thus by the implicit function theorem $\Gamma \in C^{2,\alpha}$. The constancy of P in Ω implies $P_\nu = 0$ also on Γ, i.e.

$$\left[(p-1)|u_\nu|^{p-2}u_{\nu\nu} + \frac{1}{n}\right] = 0 \quad \text{on}\Gamma.$$

Proceeding as under a) we now get

$$-1-(n-1)H|u_\nu|^{p-1}+\frac{1}{n}=0 \quad \text{or} \quad H=h(|u_\nu|) \text{ on } \Gamma.$$

But since $P\equiv c$ one may conclude that $|\nabla u|=g(u)$ for a suitable function g, and $H=h(g(\varepsilon))=const.$ on Γ. Therefore Γ has constant mean curvature and again this implies that Ω must be a ball. □

There is also an anisotropic version of the Serrin/Weinberger result, for which independent proofs were given in [**CL**] and [**WX**]. While the proof of Cianchi and Salani from Dec 2008 uses entirey different methods, the one of Wang and Xia from May 2009 follows the line of arguments given above for the Euclidean case.

Theorem 4.2. *Suppose that $\Omega\subset\mathbb{R}^n$ is a smooth connected domain, that H is a norm on $\mathbb{R}^n$ with a strictly convex unit ball, that u is a minimizer of*

$$\int_\Omega \left(\tfrac{1}{2}H(\nabla v)^2-v\right)\,dx \text{ in } W_0^{1,2}(\Omega), \quad \text{and that } H(\nabla u)=a \text{ on } \partial\Omega.$$

Then Ω is a ball in the dual norm H_0 to H of suitable radius r and

$$u(x)=\frac{r^2-H_0(x)^2}{2n}\ .$$

If we now continue the juxtaposition of Δ_p versus Δ_p^N, we should also ask ourselves about the overdetermined boundary value probem

$$\begin{cases} \qquad\qquad -\Delta_p^N u=1 & \text{in } \Omega,\\ u=0 \quad \text{and} \quad -\frac{\partial u}{\partial\nu}=a=const. & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Can we say that solutions to this problem exist only domains Ω which are balls? For general $p\in(1,2)\cup(2,\infty)$ this problem is presently unresolved, but the limiting cases $p=1$ and $p=\infty$ are also interesting.

Remark 4.3. For $p=1$ problem (4.4) degenerates into

$$|\nabla u|(n-1)H=1 \text{ in } \Omega, \qquad |\nabla u|=a \ \text{ and } u=0 \text{ on } \partial\Omega.$$

So a C^2 solution on a smooth domain satisfies $H\equiv 1/(a(n-1))$ on $\partial\Omega$. By Alexandrov's theorem Ω must then be a ball of radius $(n-1)a$.

Remark 4.4. For $p=\infty$ problem (4.4) turns into the overdetermined boundary value problem

$$-u_{\nu\nu}=1 \text{ in } \Omega, \qquad |\nabla u|=a \ \text{ and } u=0 \text{ on } \partial\Omega,$$

which can have C^1 viscosity solutions on special (non-ball) domains, e.g. stadium domains or annuli. More details on this can be found in [**BK**]. In case of a stadium domain the viscosity solution is not of class C^2, but for annuli and balls it is.

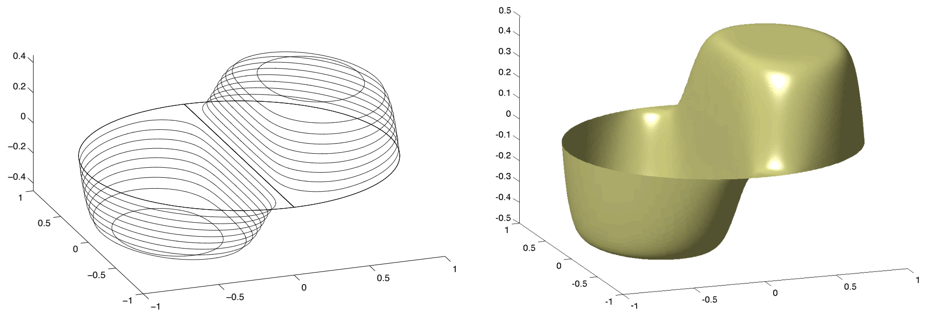

FIGURE 3. Ω a disk, $p = 1.1$, courtesy of J. Horák

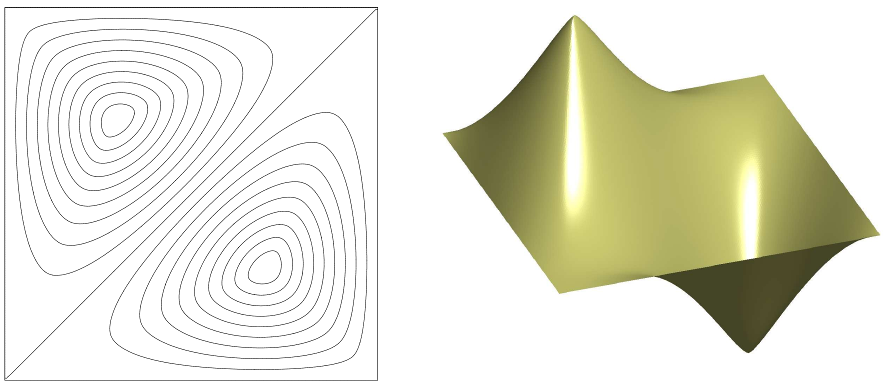

FIGURE 4. Ω a square, $p = 5$, courtesy of J. Horák

5. Open Problems

For fixed $p \in (1, \infty)$ consider the second eigenfunction to the p-Laplace operator under Dirichlet boundary conditions

$$\Delta_p u_2 + \lambda_2 |u_2|^{p-2} u_2 = 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

It changes sign, it has two nodal domains, and it can be characterized as a mountain pass going from u_1 to $-u_1$, as shown in the paper [**CFG**] of Cuesta, de Figuereido and Gossez. Clearly for $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^2$ the eigenfunction u_2 has a nodal line, and by extrapolation from the linear situation ($p = 2$) it is only natural to make **conjectures** about them.

Conjectures:

a) For Ω a disk, the nodal line of u_2 is a diameter.

b) For Ω a square the nodal line of u_2 is diagonal if $p \in (2, \infty)$
and horizontal or vertical if $p \in (1, 2)$.

There are indications that conjectures a) and b) hold for $p = 1$ in [**Par**], because for $p = 1$ the nodal line tries to minimize its length, as well as for $p = \infty$ in [**JL**], because for $p = \infty$ nodal domains of the second eigenfunction try to maximize

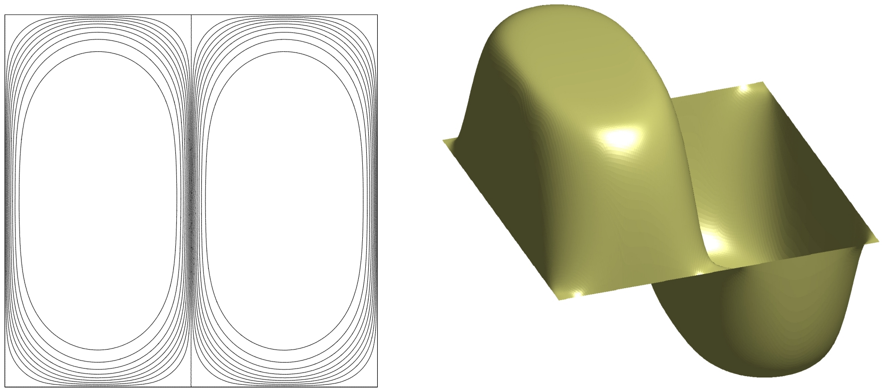

FIGURE 5. Ω a square, $p = 1.1$, courtesy of J. Horák

their inradius. Moreover, the conjectures are supported for general p by numerical evidence of Jiří Horák [**H**] in Figures 3, 4 and 5, who managed to calculate them as mountain passes according to [**CFG**].

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