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Methods and problems in discrete mathematics

Wintersemester 2019/20

— Exercise Sheet 3 —

Exercise 3.1 Consider the weight function $w \in \mathcal{C}([-1,1])$ defined by $w(t) = (1-t^2)^{-1/2}$. The corresponding orthogonal polynomials are the Chebyshev polynomials $T_n(\cos\theta) = \cos n\theta$. Verify the interlacing property of the roots of T_n.

Exercise 3.2 Let $w \in \mathcal{C}([a,b])$ be a continuous, nonnegative weight function, $w \neq 0$, and let $(p_n)_{n=0,1,\ldots}$ be a sequence of orthogonal polynomials with respect to w. Show: For each $\alpha \in \mathbb{R}$, the polynomial $p_n + \alpha p_{n-1}$ has n distinct real roots, which are interlaced with the roots of p_{n-1}.

Exercise 3.3 Consider the Chebyshev polynomials $T_n(\cos\theta) = \cos n\theta$. Let $r_1, \ldots, r_n$ be the roots of T_n.

(a) Show that the coefficients λ_i for the Gauss-Jacobi quadrature are given by the following system of linear equations:

$$\begin{pmatrix} T_0(r_1) & T_0(r_2) & \ldots & T_0(r_n) \\ T_1(r_1) & T_1(r_2) & \ldots & T_1(r_n) \\ \vdots & \vdots & \ddots & \vdots \\ T_{n-1}(r_1) & T_{n-1}(r_2) & \ldots & T_{n-1}(r_n) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} \int_{-1}^{1} T_0(t)(1-t^2)^{-1/2}\,dt \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(b) Show that there is $\lambda > 0$ such that $\lambda_i = \lambda$ for $i = 1, \ldots, n$.

(c) Determine λ.

Exercise 3.4 Present the proof of Lemma 3.8 (see page 118 in the Cohn-Kumar paper).

"Hand-in": Until Thursday **November 7**, 10 am, using the form on the course homepage.