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Methods and problems in discrete mathematics

Wintersemester 2019/20

— Exercise Sheet 5 —

Exercise 5.1 Let $S, S' \subseteq \mathbb{C}^V$ be G-invariant subspaces. Let $e_1, \dots, e_h$ be an orthonormal basis of S and let $e'_1, \dots, e'_{h'}$ be an orthonormal basis of S'. Show that the linear map

$$A : S \to S' \text{ defined by } A(e_i) = \sum_{j=1}^{h'} \langle e'_j, e_i \rangle e'_j, \text{ with } i = 1, \dots, h,$$

is a G-map.

Exercise 5.2 Decompose $\mathbb{C}^{S_3}$ as a direct orthogonal sum of irreducible unitary representations. Hint: Consider the constant function, the sign function, and the action of S_3 on the vertices of a regular triangle, which is defined by conv$\{1, e^{2\pi i/3}, e^{4\pi i/3}\}$.

Exercise 5.3 Let G be a finite group. Then $f \in \mathbb{C}^G$ defines the group determinant $\det M_f$ where

$$M_f = \left(f(xy^{-1})\right)_{x,y \in G}.$$

Show that the group determinant M_f factors as follows:

$$\det M_f = \prod_{\pi \in \widehat{G}} \det(\widehat{f}(\pi))^{d_\pi}.$$

Exercise 5.4 Use Exercise 5.3 to give formulas for the determinants of the following matrices with complex entries:

$$\begin{pmatrix} x_0 & x_{n-1} & \dots & x_2 & x_1 \\ x_1 & x_0 & \dots & x_3 & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-2} & x_{n-3} & \dots & x_0 & x_{n-1} \\ x_{n-1} & x_{n-2} & \dots & x_1 & x_0 \end{pmatrix} \qquad \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_6 & x_5 \\ x_2 & x_1 & x_5 & x_6 & x_4 & x_3 \\ x_3 & x_6 & x_1 & x_5 & x_2 & x_4 \\ x_4 & x_5 & x_6 & x_1 & x_3 & x_2 \\ x_5 & x_4 & x_2 & x_3 & x_1 & x_6 \\ x_6 & x_3 & x_4 & x_2 & x_5 & x_1 \end{pmatrix}$$

"Hand-in": Until Thursday **November 7**, 10 am, using the form on the course homepage.