Chapter II Harmonic analysis of (finite) groups

§1 Motivation: Positive-definite kernels

still have to understand: the map

$$(x, y) \mapsto P_k^n(x \cdot y) \quad \text{for } x, y \in S^{n-1}$$

is a positive (-definite) kernel.

<u>More general setup</u>

Let G be a topological group acting continuously on a topological space X.

<u>Def.</u> Let G be a group and let X be a set. We say that <u>G acts on X</u> if there is a map

$$G \times X \to X$$

which satisfies (i) $ex = x$ for all $x \in X$ [e the identity element of G]

(ii) $g\,h(x) = (gh)x$ for all $g, h \in G$, $x \in X$.

Example

$G = O(n) = \{ A \in \mathbb{R}^{n \times n} : A^T A = I_n \}$

orthogonal group

$X = S^{n-1}$ unit sphere

$$O(n) \times S^{n-1} \to S^{n-1}$$
$$(A, x) \mapsto Ax.$$

Def. Call a continuous function $K : X \times X \to \mathbb{C}$ a positive-definite kernel if for all $N \in \mathbb{N}$, $x_1, \dots, x_N \in X$ the matrix $[K(x_i, x_j)]_{1 \le i,j \le N}$ is Hermitian and positive semidefinite. We say that K is G-invariant if $K(gx, gy) = K(x, y)$ for all $g \in G$, $x, y \in X$

Example $K(x, y) = P_k^n(x \cdot y)$ is $O(n)$-invariant positive definite kernel.

Constructing / Characterising G-invariant positive-definite kernels

Def. Let V be a finite-dimensional $\mathbb{C}$-vector space with Hermitian inner product $\langle \cdot , \cdot \rangle$. We say that V is a <u>unitary representation</u> of a (finite) group G if G acts on V and the action respects the unitary structure of V:

(i) $g(\lambda v + \mu w) = \lambda g v + \mu g w$

(ii) $\langle g v, g w \rangle = \langle v, w \rangle$

for all $g \in G$, $\lambda, \mu \in \mathbb{C}$, $v, w \in V$.

In other words, there is a group homomorphism $G \to U(V)$, where $U(V)$ is the group of unitary transformations of V.

<u>Lemma</u> Let $\varphi : X \to V$ be a G-equivariant map, i.e. $\varphi(gx) = g\varphi(x)$, then the kernel

$$K(x,y) = \langle \varphi(x), \varphi(y) \rangle$$ is a positive-definite G-invariant kernel.

<u>Proof</u> · K Hermitian: ✓

· K G-invariant:

$$\begin{aligned} K(gx, gy) &= \langle \varphi(gx), \varphi(gy) \rangle \\ &= \langle g\varphi(x), g\varphi(y) \rangle \\ &= \langle \varphi(x), \varphi(y) \rangle \\ &= K(x,y). \end{aligned}$$

· K positive-definite: clear, every finite submatrix is a Gram matrix of the vectors $\varphi(x_i)$. ☒

In fact, every G-invariant positive-definite kernel arises by this construction:

Theorem Let $K: X \times X \to \mathbb{C}$ be G-invariant and positive definite. Then there is a unitary representation $(V, \langle \cdot, \cdot \rangle)$ of G and a G-equivariant map $\varphi: X \to V$ so that

$$K(x, y) = \langle \varphi(x), \varphi(y) \rangle \quad \text{for all } x, y \in X$$

holds.

Proof (the GNS construction, by Gelfand, Naimark, Segal)

Define $W = \mathbb{C}^X$ with basis vectors e_x, $x \in X$.
Define on W the positive semidefinite sesquilinear form $\langle \cdot, \cdot \rangle$ by $\langle e_x, e_y \rangle = K(x,y)$ and extend linearly. The action of G on X extends to $\mathbb{C}^X$ by $g e_x = e_{gx}$. For $v, w \in W$ we have

$$\langle gv, gw \rangle = \langle v, w \rangle \quad \text{for all } g \in G$$

because

$$\langle g e_x, g e_y \rangle = \langle e_{gx}, e_{gy} \rangle = K(gx, gy) = K(x,y)$$
$$= \langle e_x, e_y \rangle$$

Define

$$W_0 = \{ w \in \mathbb{C}^X : \langle w, w \rangle = 0 \}$$

which is a subspace of W, which is G-invariant, i.e. for $w \in W_0$, $g \in G$ we have $gw \in W_0$.

Then, the inner product on $V = W/W_0$ is positive definite and we have a unitary representation of G.

Furthermore, define $\varphi : X \to V$ by $\varphi(x) = e_x + W_0$

Then $\langle \varphi(x), \varphi(y) \rangle = \langle e_x + W_0, e_y + W_0 \rangle$

$$= \langle e_x, e_y \rangle = K(x,y).$$

$\boxtimes$

<u>Observation</u> Let $(V, \langle \cdot, \cdot \rangle)$ be a unitary representation of G. Suppose it decomposes $V = V_1 \perp V_2$, where V_1, V_2 are G-invariant.

Consider $K(x,y) = \langle \varphi(x), \varphi(y) \rangle$ with $\varphi : X \to V$.

Write $\varphi(x) = \varphi_1(x) + \varphi_2(x)$ for $\varphi_i : X \to V_i$, $i = 1, 2$.

Then,

$$\begin{aligned} K(x,y) &= \langle \varphi(x), \varphi(y) \rangle \\ &= \langle \varphi_1(x) + \varphi_2(x), \varphi_1(y) + \varphi_2(y) \rangle \\ &= \langle \varphi_1(x), \varphi_1(y) \rangle + \langle \varphi_2(x), \varphi_2(y) \rangle \\ &= K_1(x,y) + K_2(x,y). \end{aligned}$$

$\rightsquigarrow$ So: It makes sense to understand the irreducible unitary representations of G. They are the $\boxed{\text{building blocks}}$ for G-invariant positive definite kernels; the cone of G-invariant pos. def. kernels is spanned by these.

$\boxed{\text{Complete answer}}$ Peter-Weyl theorem (1927)

If G is a compact group, then every unitary rep. of G is the orthogonal direct sum of finite-dimensional irreducible unitary representations.

We shall prove the Peter-Weyl theorem for finite groups now.

§2 The Peter-Weyl theorem

Recap: discrete Fourier transform (DFT)

Consider the Abelian group $G = \mathbb{Z}/_{n}\mathbb{Z}$, $n \in \mathbb{N}$.

$a \in \mathbb{Z}/_{n}\mathbb{Z}$ defines a character

$$\chi_a : G \to \mathbb{T} = \{ z \in \mathbb{C} : z\bar{z} = 1 \}$$

$$\chi_a(x) = e^{\frac{2\pi i a x}{n}},$$

a group homomorphism from G to the torus $\mathbb{T}$.

The characters χ_a, $a \in G$, form an ONB of $\mathbb{C}^G$ with inner product $\langle f, g \rangle = \frac{1}{n} \sum_{x \in G} f(x)\overline{g(x)}$

$\hat{G} = \{ \chi_a : a \in G \}$ is called the dual group (with pointwise multiplication)

Define $\hat{f} \in \mathbb{C}^{\hat{G}}$, the DFT of f, by

$$\hat{f}(\chi) = \langle f, \chi \rangle.$$

Central properties of DFT

(i) Parseval-Plancherel formula

$$\langle f, g\rangle = (\hat{f}, \hat{g}) = \sum_{\chi\in\hat{G}} \hat{f}(\chi)\overline{\hat{g}(\chi)}$$

(ii) Fourier inversion formula

$$f(x) = \sum_{\chi\in\hat{G}} \hat{f}(\chi)\,\chi(x) \quad \text{for all } x\in G$$

(iii) Convolution identity

$$\widehat{f*g}(\chi) = \hat{f}(\chi)\cdot\hat{g}(\chi),$$

when $(f*g)(x) = \frac{1}{n}\sum_{y\in G} f(y)g(x-y)$ the convolution of $f, g\in\mathbb{C}^G$

$\rightsquigarrow$ see MPDM 2014/15, p. 97-106

Want to have the same for non-Abelian groups!

But this is impossible!

For example Fourier inversion:

$$f(xy) = \sum_{\chi \in \hat{G}} \hat{f}(\chi)\, \chi(xy)$$

$$= \sum_{\chi \in \hat{G}} \hat{f}(\chi)\, \chi(x) \cdot \chi(y)$$

$$= \sum_{\chi \in \hat{G}} \hat{f}(\chi)\, \chi(yx)$$

$$= f(yx). \qquad \text{for all } f, x, y$$

$\rightsquigarrow$ G Abelian.

<u>Problem</u> in non-Abelian groups: there are not sufficiently many characters.

<u>Solution</u> (Frobenius 1897)
Replace characters by irreducible unitary representations and derive similar formulas (but involving matrices).

Our setting

Let G be a finite group which acts transitively on set V, i.e. for all $x, y \in V$ there exists $g \in G$ so that $gx = y$.

Fix a point $e \in V$ and consider the stabiliser of e: $K = \mathrm{Stab}(G, e) = \{ g \in G \; ge = e \}$.

Then K is a subgroup of G and $V \cong G/K$.

<u>Example</u> $G = O(n)$ [ok, it is not finite, but almost as it is compact]

$$V = S^{n-1}$$

$e = (1, 0, \ldots, 0)$ the North pole

$$K \cong O(n-1)$$

$=$ all $O(n)$-matrices with 1st column e.

$$G/K \cong S^{n-1}.$$

$\mathbb{C}^V$ is a complex, finite dimensional Hilbert space with inner product $\langle f, g\rangle = \frac{1}{|V|}\sum_{x\in V} f(x)\overline{g(x)}$.

G acts on $\mathbb{C}^V$ by

$$\mu f(x) = f(\mu^{-1}x), \quad \mu\in G,\ f\in\mathbb{C}^V,\ x\in V.$$

The inner product is G-invariant:

$$\langle \mu f, \mu g\rangle = \langle f, g\rangle, \quad \mu\in G,\ f,g\in\mathbb{C}^V.$$

<u>Def.</u> Let $S, S'\subseteq \mathbb{C}^V$ be subspaces and $T: S\to S'$ be a linear map. We say

(a) S is <u>G-invariant</u> if $GS = S$

(b) $S\neq\{0\}$ is <u>G-irreducible</u> if S is G-invariant and the only G-invariant subspaces of S are $\{0\}$ and S itself

(c) T is a <u>G-map</u> if $T(\mu f) = \mu T(f)$ for all $\mu\in G$, $f\in S$, and if $\langle Tf, Tg\rangle = \langle f, g\rangle$

(d) S and S' are <u>G-equivalent</u> ($S \sim S'$) if $\exists$ G-map $T : S \to S'$ which is bijective.

<u>Two important lemmas</u>

<u>Lemma</u> (Maschke's theorem, 1899)

If S is G-invariant, $\mathcal{U} \subseteq S$ G-invariant, then
$$\mathcal{U}^\perp = \{ f \in S : \langle f, g \rangle = 0 \text{ for all } g \in \mathcal{U} \}$$
is G-invariant, too.

<u>Proof</u> For $f \in \mathcal{U}^\perp$ and $\mu \in G$ we have
$$\langle \mu f, g \rangle = \langle f, \underbrace{\mu^{-1} g}_{\in \mathcal{U}} \rangle = 0 \quad \text{for all } g \in \mathcal{U}.$$
$\boxtimes$

<u>Corollary</u> $\mathbb{C}^V$ is an orthogonal direct sum of G-irreducible subspaces.

Lemma (Schur's lemma, 1905)

Suppose S, S' are G-irreducible and $T: S \to S'$ is a G-map.

If $S \not\sim S'$, then $T = 0$.

If $S \sim S$, then either $T = 0$ or T is bijective.

Proof The kernel of T is G-invariant:

For $f \in \ker T$ and $\mu \in G$ we have

$$T(\mu f) = \mu T(f) = \mu 0 = 0.$$

Similarly, the image of T is G-invariant:

For $f \in S$ and $\mu \in G$ we have

$$\mu T(f) = T(\mu f).$$

Since S and S' are G-irreducible, we are left with one out of four possibilities:

(1) $\ker T = \{0\}$, $\operatorname{im} T = \{0\}$.

This implies $S = \{0\}$, which is impossible.

(2) $\ker T = \{0\}$, $\operatorname{im} T = S'$

Then T is bijective and $S \sim S'$.

(3) $\ker T = S$, $\operatorname{im} T = \{0\}$

Then $T = 0$.

(4) $\ker T = S$, $\operatorname{im} T = S'$.

This implies $S' = \{0\}$, which is impossible. $\boxtimes$

<u>Corollary</u> (orthogonality relation)

Let S, S' be G-irreducible, let $e_1, \dots, e_k$ be an ONB of S, $e_1', \dots, e_{k'}'$ be an ONB of S'.

If $S \not\sim S'$, then $S \perp S'$; $\langle e_i, e_j' \rangle = 0$.

If $S \sim S'$ and T is a bijective G-map with $Te_i = e_i'$, then there is a constant C so that

$$\langle e_i, e_j' \rangle = \begin{cases} C, & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

<u>Proof</u> Define the linear map $A : S \to S'$ by

$$A(e_i) = \sum_{j=1}^{h'} \langle e_j', e_i \rangle e_j'.$$

<u>Claim</u>: A is a G-map

<u>Proof</u> $\rightarrow$ Exercise.

If $S \not\sim S'$, then $A = 0$ by Schur's lemma.

Thus, $\langle e_i, e_j' \rangle = 0$.

If $S \sim S'$, then suppose $A \neq 0$. Hence, A is bijective by Schur's lemma. Consider the endomorphism

$$T^{-1}A : S \to S, \quad T^{-1}A(e_i) = \sum_{j=1}^{h} \langle e_j', e_i \rangle e_j.$$

$T^{-1}A$ is $\mathbb{C}$-linear and bijective, so it has a nonzero eigenvalue c. Corresponding eigenspace is G-invariant, so it is equal to S as S is G-irreducible. Hence, $T^{-1}A = cI$, and

$\langle e_i, e_j' \rangle = c\,\delta_{ij}$. $\boxtimes$

<u>Theorem</u> (Peter-Weyl theorem)

$$\mathbb{C}^V = \bigoplus_{k=1}^{K} H_k$$

and $$H_k = \bigoplus_{i=1}^{m_k} H_{k,i},$$

where the decomposition is orthogonal, and $H_{k,i}$ is G-irreducible, and $H_{k,i} \sim H_{k',i'}$ iff $k=k'$.

In other words, $\mathbb{C}^V$ has an orthonormal basis

$$(e_{k,i,l}), \quad k=1,\dots,K, \ i=1,\dots,m_k, \ l=1,\dots,h_k$$

so that

(1) $H_{k,i} = \mathrm{span}\{e_{k,i,1},\dots,e_{k,i,h_k}\}$ G-irreducible.

(2) $H_{k,i} \sim H_{k',i'} \iff k=k'$

(3) $\exists\ \phi_{k,i} : H_{k,1} \to H_{k,i}$ G-map,

$$\phi_{k,i}(e_{k,1,l}) = e_{k,i,l}, \quad k=1,\dots,K, \ l=1,\dots,h_k$$

<u>Proof</u> Follows from Maschke's theorem and Schur's lemma. ☒

Corollary (Bochner's theorem, 1941)

Let $K : V \times V \to \mathbb{C}$ be a positive semidefinite, Hermitian matrix which is G-invariant. Then, K can be written as

$$K(x,y) = \sum_{k=1}^{K} \sum_{i,j=1}^{m_k} f_{k,i,j} \sum_{l=1}^{h_k} e_{k,i,l}(x) \overline{e_{k,j,l}(y)}$$

$$= \sum_{k=1}^{K} \langle F_k , \overline{Z_k(x,y)} \rangle$$

where $F_k = (f_{k,i,j})_{i,j} \in \mathbb{C}^{m_k \times m_k}$ is Hermitian and positive semidefinite and where

$$(Z_k(x,y))_{ij} = \sum_{l=1}^{h_k} e_{k,i,l}(x) \overline{e_{k,j,l}(y)}$$

and where $\langle A, B \rangle = \operatorname{Tr}(B^* A)$.

Proof We have $Z_k(x,y) = Z_k(\mu x, \mu y)$ for all $\mu \in G$, $x, y \in V$ because:

$$z_k(ux, uy) = \sum_{l=1}^{h_k} e_{k,i,l}(ux)\,\overline{e_{k,j,l}(uy)}$$

$$= \sum_{l=1}^{h_k} u^{-1} e_{k,i,l}(x)\,\overline{u^{-1} e_{k,j,l}(y)}$$

$$= \sum_{l=1}^{h_k} u^{-1} \phi_{k,i}(e_{k,1,l})(x)\,\overline{u^{-1} \phi_{k,j}(e_{k,1,l})(y)}$$

$$= \sum_{l=1}^{h_k} \phi_{k,i}(u^{-1} e_{k,1,l})(x)\,\overline{\phi_{k,j}(u^{-1} e_{k,1,l})(y)}$$

We can express $u^{-1} e_{k,1,l}$ as $\sum_{r=1}^{h_k} U_{r,l}\, e_{k,1,r}$ for a unitary matrix $(U_{r,l})_{r,l=1,\dots,h_k}$. So,

$$z_k(ux, uy) = \sum_{l=1}^{h_k} \phi_{k,i}\Big(\sum_{r=1}^{h_k} U_{r,l}\, e_{k,1,r}\Big)(x) \cdot \overline{\phi_{k,j}\Big(\sum_{s=1}^{h_k} U_{s,l}\, e_{k,1,s}\Big)(y)}$$

$$= \sum_{r,s=1}^{h_k} \underbrace{\Big(\sum_{l=1}^{h_k} U_{r,l}\overline{U_{s,l}}\Big)}_{=\delta_{rs}} \phi_{k,i}(e_{k,1,r})(x)\,\overline{\phi_{k,j}(e_{k,1,s})(y)}$$

$$= \sum_{r=1}^{h_k} \phi_{k,i}(e_{k,1,r})(x)\,\overline{\phi_{k,j}(e_{k,1,r})(y)}$$

$$= \sum_{r=1}^{h_k} e_{k,i,r}(x)\,\overline{e_{k,j,r}(y)} = z_k(x, y).$$

Every Hermitian matrix $K : V \times V \to \mathbb{C}$ can be written as a linear combination of

$$(x,y) \mapsto e_{k,i,l}(x)\, \overline{e_{k',i',l'}(y)}.$$

So, $$K(x,y) = \sum_{\substack{k,k' \\ i,i' \\ l,l'}} f_{k,k',i,i',l,l'}\, e_{k,i,l}(x)\, \overline{e_{k',i',l'}(y)}.$$

To show ① $f_{k,k',i,i',l,l'} = 0$ if $k \neq k'$, $l \neq l'$

② $f_{k,k,i,i',l,l}$ does not depend on l,

so that $f_{k,i,i'} = f_{k,k,i,i',l,l}$.

For this consider the G-invariant inner product (possibly degenerate) on $\mathbb{C}^V$ defined by K

$$(f,g)_K = \frac{1}{|V|^2} \sum_{x,y \in V} K(x,y)\, f(x)\, \overline{g(y)}. \qquad (*)$$

Now the orthogonality relations imply ① & ②.

The matrices F_k are positive semidefinite

<u>because</u>:

Consider the space spanned by

$$\varphi_{k,i} = \begin{bmatrix} e_{k,i,1} \\ \vdots \\ e_{k,i,h_k} \end{bmatrix}, \quad i = 1, \dots, m_k .$$

Then $(\varphi_{k,i}, \varphi_{k,i'}) = \left(\sum_{l=1}^{h_k} e_{k,i,l}, \sum_{l=1}^{h_k} e_{k,i',l} \right)_K$

is a (possibly degenerate) inner product. Furthermore,

$$(\varphi_{k,i}, \varphi_{k,i'}) = f_{k,i,i'}.$$

$\boxtimes$

<u>Remark</u> Bochner's theorem shows that there is an isomorphism between

$(\mathcal{H}_+^V)^G$ = cone of G-invariant, Hermitian positive semidefinite matrices

and $\bigoplus_{k=1}^{K} \mathcal{H}_+^{m_k}$. In optimization this is extremely helpful, especially when m_k is small.

Back to Fourier analysis on G

Consider $\mathbb{C}^G$ with usual inner product

$$\langle f,g\rangle = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{g(x)}.$$

G acts on $\mathbb{C}^G$ by $uf(x) = f(u^{-1}x)$, aka the left regular representation of G.

Apply Peter-Weyl to $\mathbb{C}^G$:

$$\mathbb{C}^G = \bigoplus_{k=1}^{K}\bigoplus_{i=1}^{m_k} H_{k,i}\,, \qquad \dim H_{k,i} = h_k.$$

One can show: ($\to$ see literature)

① $H_{1,1},\dots,H_{K,1}$ form a complete system of representatives of irreducible unitary representations of G.

Define $\hat{G} = \{\pi : \pi \text{ irreducible unitary rep.}\}/\sim$

Then $|\hat{G}| = K$.

$$\mathbb{C}^G = \bigoplus_{\pi \in \hat{G}} \bigoplus_{i=1}^{m_k} H_{\pi,i}$$

② $h_k = m_k = d_\pi$ = degree of the irreducible unitary representation π.

So $|G| = \sum_{\pi \in \hat{G}} d_\pi^2$.

③ $|\hat{G}|$ = # conjugacy classes of G,

where x, y lie in the same conjugacy class if there is a z so that $x = z^{-1} y z$.

④ <u>(Matrix valued) Fourier transform</u> for $f \in \mathbb{C}^G$

$\hat{f}(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ for $\pi \in \hat{G}$

$$\hat{f}(\pi) = \frac{1}{|G|} \sum_{u \in G} f(u) \underbrace{\left(\langle u e_{\pi,1,j}, e_{\pi,1,i} \rangle \right)_{i,j}}_{= \pi(u)_{ij}}$$

$\pi(u) \in U(\mathbb{C}^{d_\pi})$

(i) Parseval / Plancherel formula

$$\langle f, g \rangle = \sum_{\pi \in \hat{G}} d_\pi \underbrace{\mathrm{Tr}\left(\hat{f}(\pi)\, \hat{g}(\pi)^*\right)}_{= \langle \hat{f}(\pi), \hat{g}(\pi) \rangle}$$

(ii) Fourier inversion formula

$$f(u) = \sum_{\pi \in \hat{G}} d_\pi \, \mathrm{Tr}\left(\hat{f}(\pi)\, \pi(u)^*\right)$$

(iii) Convolution identity

$$(f * g)(u) = \frac{1}{|G|} \sum_{v \in G} f(v)\, g(v^{-1}u)$$

Then

$$\widehat{f * g}(\pi) = \hat{f}(\pi)\, \hat{g}(\pi)$$

§ 3 Spherical harmonics

Apply (by analogy) § 2 to

$V = S^{n-1}$, $G = O(n)$.

Consider

$\mathcal{C}(S^{n-1}) = \{ f : S^{n-1} \to \mathbb{C} : f \text{ continuous} \}$.

Then, $O(n)$ acts on $\mathcal{C}(S^{n-1})$ by $Af(x) = f(A^{-1}x)$, and $\mathcal{C}(S^{n-1})$ has an $O(n)$-invariant inner product

$$\langle f, g \rangle = \int_{S^{n-1}} f(x) \overline{g(x)} \, d\omega(x).$$

We will see: Peter-Weyl theorem specialises here to:

Thm. $\boxed{\mathcal{C}(S^{n-1}) = \widehat{\bigoplus_{k=0}^{\infty}} H_k}$, where

H_k are $O(n)$-irreducible spaces with $H_k \simeq H_{k'} \iff k = k'$. More precisely, H_k is the space of homogeneous polynomial functions of degree k which vanish under the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$.

Bochner's theorem specialises to

<u>Thm.</u> (Schoenberg, 1942)
Every positive, $O(\mathbb{R}^n)$-invariant, real-valued, kernel $K \in \mathcal{C}(S^{n-1} \times S^{n-1})$ can be written as

$$K(x,y) = \sum_{k=0}^{\infty} f_k P_k^n(x \cdot y)$$

with $f_k \geq 0$ and $\sum_{k=0}^{\infty} f_k < \infty$, where the right hand side converges absolutely and uniformly to the left hand side.

$\left[P_k^n \text{ orthogonal w.r.t } \int_{-1}^{1} f(t) g(t) (1-t^2)^{\frac{n-3}{2}} dt \right]$

So we are in the lucky case $\boxed{m_k = 1}$ here.

Harmonic polynomials

<u>Notation</u>: • $f \in \mathbb{C}[x_1, \dots, x_n] = \mathbb{C}[x]$ polynomial

• $f \in \mathbb{C}[x]_d$ homogeneous poly. of degree d

$f(\alpha x) = \alpha^d f(x)$ for all $\alpha \in \mathbb{C}$, $x \in \mathbb{C}^n$.

- $\omega \in \mathbb{C}[x]_2$, $\quad \omega = x_1^2 + \cdots + x_n^2$

- <u>Nabla operator</u>

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)^T$$

- <u>Laplace operator</u>

$$\Delta = \nabla^T \nabla = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} = \omega(\nabla)$$

$$\Delta : \mathbb{C}[x]_d \to \mathbb{C}[x]_{d-2}$$

- inner product on $\mathbb{C}[x]_d$

$$\langle f, g \rangle = f(\nabla)\, \overline{g}$$

easy to see: monomials are orthogonal with

$$\langle x_1^{m_1} \cdots x_n^{m_n}, x_1^{m_1} \cdots x_n^{m_n} \rangle = m_1! \cdots m_n!$$

$\rightsquigarrow$ nice interpretation: polynomials as symmetric tensors.
($\rightsquigarrow$ literature).

<u>Lemma</u> $\langle \omega f, g \rangle = \langle f, \Delta g \rangle$ $\quad$ [ω is adjoint of Δ]

for $f \in \mathbb{C}[x]_{d-2}$, $g \in \mathbb{C}[x]_d$.

<u>Proof</u> $f \mapsto f(\nabla)$ is an algebra isomorphism between $\mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$. So,

$$\langle \omega f, g \rangle = \frac{1}{d!} \omega f(\nabla) \overline{g} = \frac{1}{d!} \omega(\nabla) f(\nabla) \overline{g}$$
$$= \frac{1}{d!} f(\nabla) \overline{\Delta g}$$
$$= \langle f, \Delta g \rangle.$$

$\boxtimes$

<u>Def.</u> $\mathrm{Harm}_d = \{ f \in \mathbb{C}[x]_d : \Delta f = 0 \}$

<u>harmonic polynomials</u>

<u>Thm.</u> $\mathbb{C}[x]_d = \mathrm{Harm}_d \perp \omega \,\mathrm{Harm}_{d-2} \perp \omega^2 \mathrm{Harm}_{d-4} \perp \ldots,$

$\dim \mathrm{Harm}_k = \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}.$

<u>Proof</u> ① $\Delta : \mathbb{C}[x]_d \to \mathbb{C}[x]_{d-2}$ is surjective

Let $f \in (\Delta(\mathbb{C}[x]_d))^\perp$. Then for $g \in \mathbb{C}[x]_d$ we have

$$0 = \langle f, \Delta g \rangle = \langle \omega f, g \rangle.$$

So $\omega f = 0$ and hence $f = 0$.

(2) Dimension formula for linear maps give

$$\dim \mathrm{Harm}_k = \dim \mathbb{C}[x]_k - \dim \mathbb{C}[x]_{k-2}$$
$$= \binom{n+k-1}{n-1} + \binom{n+k-3}{n-1}.$$

(3) $\mathbb{C}[x]_d = \mathrm{Harm}_d \perp \omega\, \mathbb{C}[x]_{d-2}$

for $f \in \mathrm{Harm}_d$ and $g \in \mathbb{C}[x]_{d-2}$ we have

$$\langle f, \omega g \rangle = \langle \nabla f, g \rangle = 0$$

(4) Now the statement follows by induction, we only have to verify that $\omega\, \mathrm{Harm}_{d-2} \perp \omega^2\, \mathbb{C}[x]_{d-4}$.

<u>Lemma</u> $\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} p = d\, p$ for $p \in \mathbb{C}[x]_d$

<u>Proof</u> Check for monomial basis, $p = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ with $d = \alpha_1 + \alpha_2 + \dots + \alpha_n$. Then

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} (x_1^{\alpha_1} \dots x_n^{\alpha_n}) = \sum_{i=1}^{n} \alpha_i\, x_1^{\alpha_1} \dots x_n^{\alpha_n} = d\, x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

[Special case of Euler's homogeneous function theorem]

Lemma $\Delta \omega p = (4d+2n)p + \omega \Delta p$ for $p \in \mathbb{C}[x]_d$.

Proof We have

$$\frac{\partial^2}{\partial x_i^2} x_i^2 p = x_i^2 \frac{\partial}{\partial x_i^2} p + 4 x_i \frac{\partial}{\partial x_i} p + 2p,$$

and so

$$\begin{aligned}
\Delta \omega p &= \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\Big(\sum_{j=1}^{n} x_j^2\Big) p \\
&= \sum_{i=1}^{n}\Big(x_i^2 \frac{\partial}{\partial x_i^2} p + 4 x_i \frac{\partial}{\partial x_i} p + 2p\Big) \\
&\qquad + \sum_{i \neq j} x_j^2 \frac{\partial^2}{\partial x_i^2} p \\
&= \omega \Delta p + 2np + 4 \underbrace{\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} p}_{= dp \text{ (by prev. lemma)}} .
\end{aligned}$$

$\boxtimes$

Back to $\omega \mathrm{Harm}_{d-2} \perp \omega^2 \mathbb{C}[x]_{d-4}$. For $f \in \mathrm{Harm}_{d-2}$ and $g \in \mathbb{C}[x]_{d-4}$ we have

$$\begin{aligned}
\langle \omega f, \omega^2 g \rangle &= \langle f, \Delta \omega^2 g \rangle \\
&= \langle f, (4d+2n)\omega g + \omega \Delta \omega g \rangle \\
&= \langle \underbrace{f}_{\mathrm{Harm}_{d-2}}, \underbrace{\omega((4d+2n)g + \Delta \omega g)}_{\omega \cdot \mathbb{C}[x]_{d-4}} \rangle = 0 \quad (2)
\end{aligned}$$

$\boxtimes$

The space Harm_d is invariant under the action of $O(n)$ given by $Af(x) = f(A^{-1}x)$, because

$$\Delta(Af) = A\Delta f.$$

Also $\omega\, \mathbb{C}[x]_{d-2}$ is $O(n)$-invariant.

<u>Def.</u> The space of <u>spherical harmonics</u> of degree k is

$$H_k = \{ f|_{S^{n-1}} : f \in \mathrm{Harm}_k \}$$

↑ restriction of f to S^{n-1}

The spaces Harm_k and H_k are equivalent $O(n)$-invariant spaces. Bijections:

$$\varphi : \mathrm{Harm}_k \to H_k, \quad \varphi(f) = f|_{S^{n-1}}$$

$$\varphi^{-1} : H_k \to \mathrm{Harm}_k, \quad \varphi^{-1}(f)(x) = \|x\|^k f\left(\frac{x}{\|x\|}\right)$$

<u>Remark</u> Name "spherical harmonics" was introduced by Thomson (Lord Kelvin) and Tait (1867).

Def. Let $e_1, \dots, e_{h_k}$ be an ONB of H_k. Define the zonal spherical function

$$z_k : S^{n-1} \times S^{n-1} \to \mathbb{C} \quad \text{by} \quad z_k(x,y) = \sum_{i=1}^{h_k} e_i(x)\overline{e_i(y)}.$$

Remark Same construction as in Bochner's Theorem.

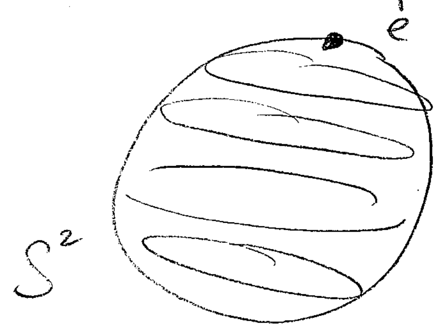
Lemma (a) z_k does not depend on the choice of ONB.

(b) $z_k(Ax, Ay) = z_k(x,y)$ for all $A \in O(n)$, $x,y \in S^{n-1}$

Proof $\rightsquigarrow$ see proof of Bochner's theorem. $\boxtimes$

Def. Let $e \in S^{n-1}$ be a point. Consider the stabiliser subgroup K of e in $O(n)$; $K = \{A \in O(n) : Ae = e\}$. We say that a function $f : S^{n-1} \to \mathbb{C}$ is a zonal spherical function with pole e if $Af = f$ for all $A \in K$.

Example $x \mapsto z_k(x,e)$ is a zonal spherical function with pole e.



function value only depends on latitude.

Lemma Let $\mathbb{C}[S^{n-1}]_d$ be the space of homogeneous polynomial function on S^{n-1} of degree d. Let $U \subseteq \mathbb{C}[S^{n-1}]_d$ be an $O(n)$-invariant subspace. If the dimension of all zonal spherical functions with pole e in U is exactly one, then U is $O(n)$-irreducible.

Proof Suppose U is not irreducible and splits as $U = V \perp V^{\perp}$. Let $e_1, \dots, e_m$ be an ONB of V and $f_1, \dots, f_n$ one of $V^{\perp}$. Then

$$z_V(x) = \sum_{i=1}^{m} e_i(e)\overline{e_i(x)}$$

$$z_{V^\perp}(x) = \sum_{i=1}^{n} f_i(e)\overline{f_i(x)}$$

are linearly independent zonal spherical functions with pole e. $\boxtimes$

Theorem (Peter-Weyl for S^{n-1})

Let $\mathbb{C}[S^{n-1}]_{\leq d}$ be the space of poly. function on S^{n-1} of degree $\leq d$. Then

$$\boxed{\mathbb{C}[S^{n-1}]_{\leq d} = H_0 \perp H_1 \perp \dots \perp H_d}$$

where H_k are $O(n)$-irreducible and pairwise inequivalent.

Proof Apply the previous lemma: Let $f \in H_k$ be a zonal spherical function with pole $e = (1, 0, \dots, 0)^T$. Consider f as polynomial in Harm_k. Write

$$f(x_1, \dots, x_n) = \sum_{i=0}^{k} x_1^{k-i} f_i(x_2, \dots, x_n).$$

The f_i's are homogeneous of degree i. Since f is invariant under K, the f_i's only depend on the norm of $(x_2, \dots, x_n)$. So

$$f_i(x_2, \dots, x_n) = c_i \left(x_2^2 + \dots + x_n^2\right)^{i/2} \quad \text{for } c_i \in \mathbb{C}.$$

In particular, $f_i = 0$ if i odd.

Together

$$f(x_1, \dots x_n) = \sum_{i=0}^{k/2} c_i x_1^{k-2i} \left(x_2^2 + \dots + x_n^2\right)^i.$$

Since f is harmonic:

$$0 = \Delta f = \sum_{i=1}^{k/2} \left(\alpha_i c_i + \beta_i c_{i-1}\right) x_1^{k-2i} \left(x_2^2 + \dots + x_n^2\right)^{i-1}$$

with $\alpha_i = 2i(n+2i-3)$, $\beta_i = (k-2i+1)(k-2i+2)$ (direct verification).

So

$$\alpha_1 c_1 + \beta_1 c_0 = 0 \quad \Rightarrow \quad c_1 = -\frac{\beta_1}{\alpha_1} c_0$$

$$\alpha_2 c_2 + \beta_2 c_1 = 0 \quad \Rightarrow \quad c_2 = -\frac{\beta_2}{\alpha_2} c_1 = \frac{\beta_2 \beta_1}{\alpha_2 \alpha_1} c_0$$

$$\vdots$$

$$\Rightarrow \quad c_i = (-1)^i \frac{\beta_i \cdots \beta_0}{\alpha_i \cdots \alpha_0} c_0, \quad i = 1, \dots, k/2.$$

So f is only determined by c_0. Thus, the space of zonal spherical functions with pole e in H_k has dimension 1, so H_k is $O(n)$-irreducible

For $n > 2$, the dimensions h_k all differ. So the H_k's are pairwise inequivalent. Case $n = 2$: $\rightarrow$ Exercise.

$\boxtimes$

<u>Remark</u> Dimensions of H_k $\left(\binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}\right)$

<u>$n = 2$</u>: constant 2 for $k \geq 1$.

<u>$n = 3$</u>: $2k+1$ $\rightsquigarrow$ physics.

<u>$n = 4$</u>: $(k+1)^2$

Addition formula

<u>Goal</u> Compute $Z_k(x,y) = \sum_{i=1}^{h_k} e_{k,i}(x)\,\overline{e_{k,i}(y)}$ explicitly.

<u>Theorem</u> There is a positive constant α_k so that

$$\boxed{Z_k(x,y) = \alpha_k P_k^n(x\cdot y).}$$

In particular, this implies the <u>positivity property</u>

$(x,y) \mapsto P_k^n(x\cdot y)$ is a positive definite $O(n)$-invariant kernel.

<u>Proof</u> Let $z_k \in H_k$ and $z_{k'} \in H_{k'}$ be zonal spherical functions with pole $e=(1,0,\dots,0)^T$. By the orthogonality relations we have for $k \neq k'$:

$$(z_k, z_{k'}) = \int_{S^{n-1}} z_k(x)\,\overline{z_{k'}(x)}\,d\omega(x) = 0.$$

Rewrite this using spherical coordinates and use the fact that $z_k, z_{k'}$ are K-invariant:

$$(z_k, z_{k'}) = \int_{-1}^{1} z_k(x)\,\overline{z_k'(x)}\,(1-x_1^2)^{\frac{n-3}{2}}\,dx_1 = 0.$$

Hence, Z_k is a multiple of P_k^n:

$$Z_k(x_1, x_2, \ldots, x_n) = \alpha_k \, P_k^n(x_1).$$

So we have the <u>addition formula</u>

$$\sum_{i=1}^{h_k} e_i(x)\,\overline{e_i(y)} = \alpha_k \, P_k^n(x \cdot y).$$

Since $P_k^n(1) = 1$, we have $\alpha_k > 0$. $\boxtimes$

<u>Spherical t-designs</u>

<u>Recall</u> $\mathcal{C} \subseteq S^{n-1}$ is a spherical t-design if for all polynomials f of deg. $\leq t$:

$$\frac{1}{|\mathcal{C}|} \sum_{x \in \mathcal{C}} f(x) = \int_{S^{n-1}} f(x)\, d\omega(x).$$

<u>Theorem</u> (Spherical design strength test).
$\mathcal{C} \subseteq S^{n-1}$ spherical M-design $\iff \sum_{x,y \in \mathcal{C}} P_k^n(x \cdot y) = 0$ for $k = 1, \ldots, M$.

<u>Proof</u> „$\Rightarrow$": For $k=1,\dots,M$, $i=1,\dots,h_k$ consider $e_{k,i}$ ONB for H_k. Since $\mathcal{C}$ is a spherical M-design we have

$$\frac{1}{|\mathcal{C}|}\sum_{x\in\mathcal{C}} e_{k,i}(x) = \int_{S^{n-1}} e_{k,i}(x)\,d\omega(x) = 0$$

because $H_k \perp H_0$ and $H_0 = \mathbb{C}\,1$.

So,

$$0 = \sum_{i=1}^{h_k}\Big(\sum_{x\in\mathcal{C}} e_{k,i}(x)\Big)\Big(\sum_{y\in\mathcal{C}}\overline{e_{k,i}(y)}\Big) = \alpha_k \sum_{x,y\in\mathcal{C}} P_k^n(x\cdot y).$$

<u>„$\Leftarrow$"</u>: Let f be a polynomial of degree $\le M$. Consider f as a polynomial function on S^{n-1} and expand $f(x) = \sum_{k=0}^{M}\sum_{i=1}^{h_k} a_{k,i}\, e_{k,i}(x)$.

<u>Claim</u> $\frac{1}{|\mathcal{C}|}\sum_{x\in\mathcal{C}} f(x) \overset{\circledast}{=} \frac{1}{|\mathcal{C}|}\sum_{x\in\mathcal{C}} a_{0,1} = a_{0,1} = \int_{S^{n-1}} f(x)\,d\omega(x).$

Proof of ⊛

It suffices to show

$$\sum_{x \in \mathcal{C}} e_{k,i}(x) = 0 \quad \text{for } k = 1, \dots, M, \; i = 1, \dots, h_k.$$

This follows from an easy sum-of-squares argument. By assumption we know

$$0 = \sum_{x,y \in \mathcal{C}} P_k^n(x \cdot y) = \frac{1}{d_k} \sum_{x,y \in \mathcal{C}} \sum_{i=1}^{h_k} e_{k,i}(x) \overline{e_{k,i}(y)}$$

$$= \frac{1}{d_k} \sum_{i=1}^{h_k} \underbrace{\left(\sum_{x \in \mathcal{C}} e_{k,i}(x) \right) \left(\overline{\sum_{x \in \mathcal{C}} e_{k,i}(x)} \right)}_{\geq 0}.$$

Hence $\sum_{x \in \mathcal{C}} e_{k,i}(x) = 0$. $\boxtimes$

Remark For given n, t it is interesting to find spherical t-design $\mathcal{C} \subseteq S^{n-1}$ with $|\mathcal{C}|$ as small as possible. An LP bound ($\to$ Exercise) gives lower bounds.

Delsarte, Goethals, Seidel (1977) showed

$$N(n,t) \geq \begin{cases} \binom{n-1+k}{n-1} + \binom{n-1+k-1}{n-1} & \text{for } t = 2k \\ 2\binom{n-1+k}{n-1} & \text{for } t = 2k+1. \end{cases}$$

Asymptotically

$$N(n,t) \in \Omega(t^{n-1}).$$

Bondarenko, Radchenko, Viazovska (2013) proved

$$N(n,t) \in O(t^{n-1}).$$

using tools from topology (Brouwer degree theory).

* * * * *