Universität zu Köln
Mathematisches Institut
Prof. Dr. F. Vallentin
Dr. A. Gundert
A. Heimendahl

Methods and problems in discrete mathematics

Wintersemester 2019/20

— Exercise Sheet 8 —

Exercise 8.1 Let $G = (V, E)$ be a connected, k-regular graph with n vertices and let

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$

be the eigenvalues of the adjacency matrix of G. Show that the following three statements are equivalent:

(a) G is bipartite,

(b) $\lambda_i = -\lambda_{n-i}$ for $i = 1, \ldots, n$,

(c) $\lambda_n = -k$.

Exercise 8.2 The k-th power of a graph G, denoted by G^k, is a graph with the same vertex set as G and two vertices u, v are adjacent in G^k if and only if there is a path from u to v with at most k edges. Show that for fixed k the family of k-th graph powers of cycle graphs C_n^k is *not* a family of expanders.

Exercise 8.3 Consider the additive group $G = (\mathbb{Z}/2\mathbb{Z})^n = \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$. The cube graph Q_n has vertex set G and two vertices $x, y \in G$ are adjacent if and only if their sum $x + y \in G$ has exactly one non-zero coordinate. Compute the spectral gap of Q_n.

Exercise 8.4 Let $G = (V, E)$ be a k-regular graph. Show that for any $U \subseteq V$ the inequality

$$\left| |\delta(U)| - \frac{k|U||V \setminus U|}{|V|} \right| \leq \lambda_2 \sqrt{|U||V \setminus U|}$$

holds.

"Hand-in": Until Thursday **December 12**, 10 am, using the form on the course homepage.