Universität zu Köln
Mathematisches Institut
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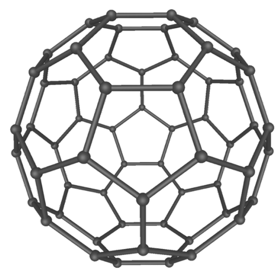
Methods and problems in discrete mathematics

Wintersemester 2019/20

— Exercise Sheet 9 —

Exercise 9.1

(a) Numerically compute an eigenvector g for the second smallest eigenvalue $3 - \lambda_2$ of the Laplacian of the vertex-edge graph of the truncated icosahedron (football):



(from https://en.wikipedia.org/wiki/Fullerene)

(b) Use g to find (e.g., by computer) a set $U \subseteq V$ with $\delta(U)/|U| \leq \sqrt{6(3 - \lambda_2)}$.

(c) Can you also compute the expansion of this graph?

Exercise 9.2 Let G be a k-regular graph on n vertices with adjacency matrix A.

(a) Show that $\mathrm{Tr}(A^2) \geq nk$.

(b) Using (a), show that $\lambda_2(A) \geq \sqrt{k \cdot \frac{n-k}{n-1}}$.

Exercise 9.3 Let G be a simple, k-regular graph and let $A_r \in \mathbb{R}^{V \times V}$ as in the lecture:

$$(A_r)_{i,j} = \text{number of paths } (i, x_1, x_2, \ldots, x_{r-1}, j) \text{ without backtracking.}$$

Show that $A_1 A_r = A_r A_1 = A_{r+1} + (k-1)A_{r-1}$ for $r \geq 2$.

Exercise 9.4 For a non-singular 2×2-matrix A over $\mathbb{Z}_n$, $b \in \mathbb{Z}_n^2$ and $f \in \mathbb{C}^{\mathbb{Z}_n^2}$, let $g \in \mathbb{C}^{\mathbb{Z}_n^2}$ with $g(x) = f(Ax + b)$. Show:

$$\hat{g}(y) = e^{-\frac{2\pi i \langle A^{-1}b, y\rangle}{n}} \hat{f}((A^{-1})^{\mathsf{T}} y).$$

"Hand-in": Until Thursday **December 19**, 10 am, using the form on the course homepage.