Chapter III Expander graphs

§1 Definitions and basic properties

§2 Cheeger's inequality

§3 The bound of Alon and Boppana

§4 The Margulis construction

Graphs in general and expander graphs in particular:

- prime objects of study in discrete mathematics
- expander graphs are simultaneously sparse (few edges) and highly connected
- Very useful in computer science, pure and applied mathematics:
  - design and analysis of communication networks
  - error correcting codes; pseudorandomness
  - convergence rates of Markov chains
  - metric embeddings (→ Chapter IV)
  - ...

§ 1 Introduction and basic properties

Let $G = (V, E)$ be a graph. Encode G by the <u>adjacency matrix</u> $A \in \mathbb{R}^{V \times V}$ where

$A = (A_{ij})_{i,j \in V}$ and A_{ij} = number of edges joining vertex i and vertex j.

<u>Example</u>

$$A = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

We say G is <u>simple</u> if $A_{ij} \in \{0,1\}$ for all $i, j \in V$,

G has <u>no loops</u> if $A_{ii} = 0$ for all $i \in V$,

G is <u>k-regular</u> if $\sum_{j \in V} A_{ij} = k$ for all $i \in V$.

(if G has no loops, then every vertex in G has exactly k neighbors).

Since A is real and symmetric, it has a <u>spectral decomposition</u>

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^T, \quad \text{where } n = |V|$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are the (real) eigenvalues of A and $u_1, \dots, u_n \in \mathbb{R}^n$ form an ONB of corresponding eigenvectors, $Au_i = \lambda u_i$, $i = 1, \dots, n$.

<u>Theorem</u> Let G be k-regular. Then,

(a) $\lambda_1 = k$

(b) $|\lambda_i| \leq k$ for $i = 2, \dots, n$

(c) The multiplicity of λ_1 equals the number of connected components of G.

<u>Proof</u>

<u>(a) & (b)</u>: The all-ones vector $e = (1, \dots, 1)^T$ is an eigenvector of A with eigenvalue k.

Let λ be an eigenvalue of A with eigenvector $x \in \mathbb{R}^V$, $Ax = \lambda x$.

Let $i \in V$ be such that

$$|x_i| = \max_{j \in V} |x_j|.$$

We may assume $x_i > 0$ (change x by $-x$ if $x_i < 0$).

Then,

$$|\lambda| x_i = |\lambda x_i| = \Big| \sum_{j \in V} A_{ij} x_j \Big| \le \sum_{j \in V} A_{ij} |x_j|$$

$$\le \underbrace{\sum_{j \in V} A_{ij}}_{= k} x_i \quad \Rightarrow \quad |\lambda| \le k.$$

(c) Here we only show: If G is connected, then the multiplicity of λ_1 equals 1.
(general statement $\rightsquigarrow$ Exercise).

Let x be an eigenvector with eigenvalue k.

to show: x is a multiple of $e = (1, \dots, 1)^T$

Again, choose $i \in V$ s.t. $|x_i| = \max_{j \in V} |x_j|$.

We have

$$x_i = \left(\tfrac{1}{k} A x\right)_i = \frac{1}{k} \sum_{j \in V} A_{ij} x_j = \sum_{j \in V} \frac{A_{ij}}{k} x_j$$

So x_i is a convex combination of x_j, $j \in V$.
Since $|x_i| \geq |x_j|$, it follows that $x_i = x_j$ for all $j \in N_i = \{ l \in V : A_{il} \neq 0 \}$ neighbors of i.

Because G is connected, this property propagates to all vertices. So, x and e are linearly dependent.

__Def.__ The difference $\lambda_1 - \lambda_2$ is called the __spectral gap__ of G

__Corollary__ A k-regular graph is connected iff its spectral gap is __positive__.

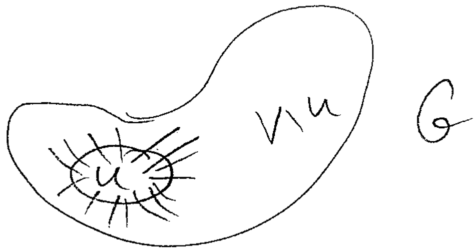
__Will see:__ Spectral gap measures "connectedness" of a graph.

Combinatorial characterisation of "connectedness"

Def. Let $G=(V,E)$ be a graph. The Cheeger constant (edge expansion ratio) of G is defined as

$$h(G) = \min_{\substack{U \subseteq V,\\ U \neq \emptyset, V}} \frac{|\delta(U)|}{\min\{|U|, |V \setminus U|\}},$$

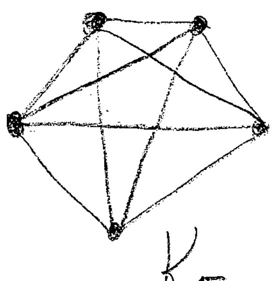
where $\delta(U) = \{e \in E : e = \{i,j\}, i \in U, j \in V \setminus U\}$ is the cut defined by U.



Note: $\delta(U) = \delta(V \setminus U)$.

Intuition If G is highly connected, then $h(G)$ is large.

Example (a) Complete graph K_n on n vertices.

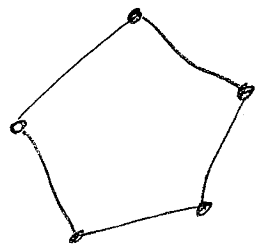


For $U \subseteq V$ with $|U| = \ell$ we have

$|\delta(U)| = \ell(n-\ell)$, so

$h(G) \geq \frac{\ell(n-\ell)}{\ell} = n-\ell \underset{n\to\infty}{\longrightarrow} \frac{n}{2}$

(b) Cycle graph C_n



C_5

Can show ($\to$ Exercise):

$$\lim_{n\to\infty} h(C_n) = 0.$$

(a) is very connected but has (too) many edges,
(b) is few edges but is not very connected.

Def. Let $(G_n)_{n\in\mathbb{N}}$, $G_n = (V_n, E_n)$, be a family of k-regular graphs. It is called a <u>family of expanders</u> if $|V_n| \to \infty$ for $n \to \infty$ and if there is $\varepsilon > 0$ such that $h(G_n) \geq \varepsilon$ for all $n \in \mathbb{N}$.

<u>Questions</u>:
- How to recognise an expander?
  [Computing $h(G)$ is computationally difficult]
  $\rightsquigarrow$ use spectral gap (§2)
- How big can the spectral gap be?
  $\rightsquigarrow$ Alon-Boppana (§3)
- Do expanders exist? $\rightsquigarrow$ Construction of Margulis (§4)

§ 2 Cheeger's inequality

Known Computing $h(G)$ is NP hard.

Can use spectral gap to approximate $h(G)$.

Large spectral gap implies high expansion

Thm. Let $G = (V, E)$ be a connected k-regular graph without loops. Then

$$\boxed{\frac{k - \lambda_2}{2} \leq h(G)}.$$

Proof Choose an arbitrary orientation of the edges, so every edge $e \in E$ has a source vertex e^- and a target vertex e^+ : $e^- \xrightarrow{e} e^+$.

Define the simplicial coboundary operator

$$d : \mathbb{C}^V \to \mathbb{C}^E$$

by

$$d f(e) = f(e^+) - f(e^-).$$

[This is essentially the incidence matrix $M^T \in \{-1, 0, +1\}^{E \times V}$ of a digraph]

$\mathbb{C}^V$ and $\mathbb{C}^E$ are complex Hilbert space with inner products e.g.

$$(f,g) = \sum_{x\in V} f(x)\overline{g(x)} \quad \text{for } f,g \in \mathbb{C}^V.$$

d has an adjoint operator $d^* : \mathbb{C}^E \to \mathbb{C}^V$ satisfying

$$(df, g)_{\mathbb{C}^E} = (f, d^* g)_{\mathbb{C}^V} \qquad f\in\mathbb{C}^V, g\in\mathbb{C}^E.$$

Define the incidence matrix $M \in \{-1,0,+1\}^{V\times E}$ by $M(x,e) = \begin{cases} +1, & \text{if } x = e^+ \\ -1, & \text{if } x = e^- \\ 0, & \text{otherwise} \end{cases}$

Then

$$df(e) = \sum_{x\in V} M(x,e) f(x) = (M^T f)(e)$$

and

$$d^* g(x) = \sum_{e\in E} M(x,e) g(e) = (Mg)(x).$$

Now define the combinatorial Laplace operator

$\Delta = d^* d : \mathbb{C}^V \to \mathbb{C}^V$ and check that

$$\Delta = kI - A, \qquad A \text{ adjacency matrix of } G.$$

Note that Δ does not depend on chosen orientation.

The eigenvalue of Δ are $0, k-\lambda_2, \dots, k-\lambda_n \geq 0$

By the Rayleigh-Ritz principle we have

$$k - \lambda_2 = \max_{\substack{f \in \mathbb{C}^V \setminus \{0\} \\ (f,e)=0}} \frac{(\Delta f, f)}{(f,f)}, \qquad e = (1, \dots, 1)^T \in \mathbb{C}^V,$$

so for $f \in \mathbb{C}^V$ with $0 = (f,e) = \sum_{x \in V} f(x)$ we have

$$\|df\|^2 = (df, df)$$

$$= (\Delta f, f) \geq (k-\lambda_2)(f,f).$$

Choose f as follows: Fix $F \subseteq V$ and set

$$f(x) = \begin{cases} |V \setminus F| & \text{if } x \in F \\ -|F| & \text{if } x \notin F. \end{cases}$$

Then:

- $\sum_{x\in V} f(x) = 0$

- $\|f\|^2 = (f,f) = |F|\,|V\setminus F|^2 + |V\setminus F|\,|F|^2$

$= |F|\,|V\setminus F|\,|V|.$

- $df(e) = \begin{cases} 0 & \text{if } e\notin \delta(F), \\ \pm |V| & \text{if } e\in \delta(F). \end{cases}$

- $\|df\|^2 = |V|^2 |\delta(F)|.$

Together

$|V|^2|\delta(F)| \geq (k-\lambda_2)\ |F|\cdot|V\setminus F|\cdot|V|$

$\Rightarrow \quad \frac{|\delta(F)|}{|F|} \geq (k-\lambda_2)\ \frac{|V\setminus F|}{|V|}$

If $|F| \leq \frac{|V|}{2}$, we get $\frac{|\delta(F)|}{|F|} \geq (k-\lambda_2)/2.$

Hence $h(G) \geq \frac{k-\lambda_2}{2}.$ $\boxtimes$

High expansion implies large spectral gap

Thm. ("Cheeger's inequality")

Let $G=(V,E)$ be a connected k-regular graph without loops. Then

$$\boxed{h(G) \leq \sqrt{2k(k-\lambda_2)}\,.}$$

Remark This is a discrete analogue of a famous inequality in differential geometry due to Cheeger (1970); found by Dodziuk (1984).

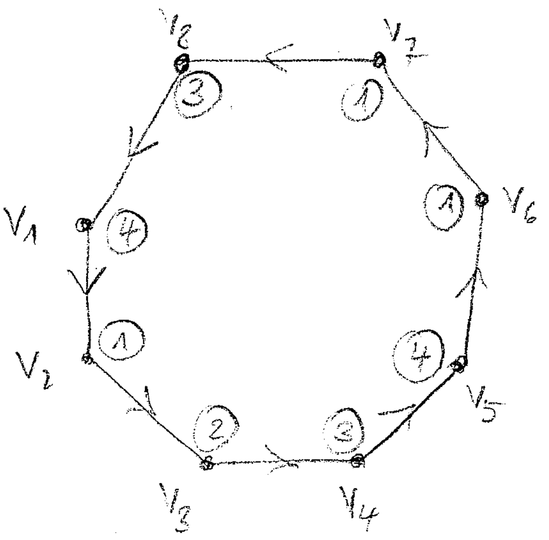
Proof Let $f \in \mathbb{C}^V$ be real-valued and nonnegative, $f(x) \geq 0$ for all $x \in V$.

Define
$$B_f = \sum_{e \in E} |f(e^+)^2 - f(e^-)^2|.$$

Denote by $\beta_r > \beta_{r-1} > \dots > \beta_1 > \beta_0 \geq 0$ the values of f. Define the level sets

$$L_i = \{x \in V : f(x) \geq \beta_i\}, \quad i = 0, 1, \dots, r.$$

Example C_8



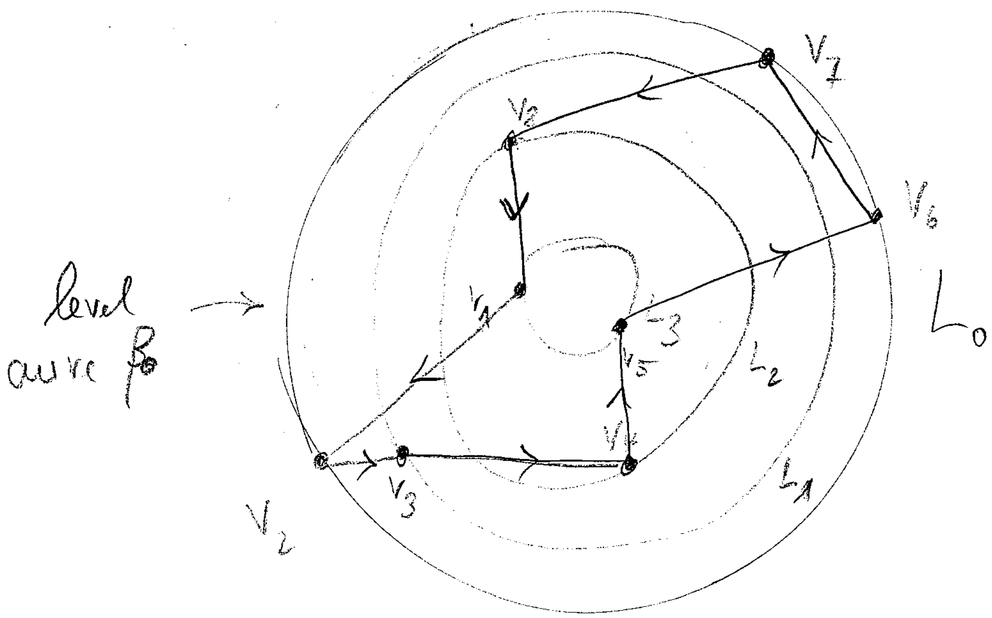
f-values in O

$$\beta_3 > \beta_2 > \beta_1 > \beta_0$$

$$\begin{array}{cccc} 4 & 3 & 2 & 1 \end{array}$$

$L_0 = \{v_1, \dots v_8\}$, $\quad \delta(L_0) = \emptyset$

$L_1 = \{v_1, v_3, v_4, v_5, v_8\}$, $\quad \delta(L_1) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_5, v_6\}, \{v_6, v_7\}\}$, $\quad |\delta(L_1)| = 4$



Step 1 $\mathcal{B}_f = \sum_{i=1}^{r} |\delta(L_i)| (\beta_i^2 - \beta_{i-1}^2)$

Proof Set $E_f = \{e \in E : f(e^+) \neq f(e^-)\}$. Then

$$\mathcal{B}_f = \sum_{e \in E_f} | f(e^+)^2 - f(e^-)^2 |.$$

Edge $e \in E_f$ connects some vertex x with $f(x) = \beta_{i(e)}$ to some vertex y with $f(y) = \beta_{j(e)}$. We may assume $i(e) > j(e)$ and so $\beta_{i(e)} > \beta_{j(e)}$. So,

$$\begin{aligned}
\mathcal{B}_f &= \sum_{e \in E_f} (\beta_{i(e)}^2 - \beta_{j(e)}^2) \\
&= \sum_{e \in E_f} (\beta_{i(e)}^2 - \beta_{i(e)-1}^2 + \beta_{i(e)-1}^2 - \cdots - \beta_{j(e)+1}^2 + \beta_{j(e)+1}^2 - \beta_{j(e)}^2) \\
&= \sum_{e \in E_f} \sum_{l=j(e)+1}^{i(e)} (\beta_l^2 - \beta_{l-1}^2). \\
&= \sum_{i=1}^{r} |\delta(L_i)| (\beta_i^2 - \beta_{i-1}^2)
\end{aligned}$$

meditation about the diagram in the example: every edge $e \in E_f$ connecting x with $f(x) = \beta_{i(e)}$ and y with $f(y) = \beta_{j(e)}$ crosses every level curve β_l between $\beta_{i(e)}$ and $\beta_{j(e)}$.

Step 2 $B_f \le \sqrt{2k}\, \|df\|\, \|f\|$

Proof

$$B_f = \sum_{e\in E} |f(e^+)+f(e^-)| \cdot |f(e^+)-f(e^-)|$$

$$\le \left(\sum_{e\in E} (f(e^+)+f(e^-))^2\right)^{1/2} \left(\sum_{e\in E} (f(e^+)-f(e^-))^2\right)^{1/2}$$

$\uparrow$ Cauchy-Schwarz inequality $|\langle v,w\rangle| \le \|v\|\cdot\|w\|$

$$\le \sqrt{2}\left(\sum_{e\in E} f(e^+)^2+f(e^-)^2\right)^{1/2} \|df\|$$

$\uparrow$ $(a+b)^2 \le 2(a^2+b^2)$

$$= \sqrt{2k}\left(\sum_{x\in V} f(x)^2\right)^{1/2} \|df\|.$$

Step 3 The support of f is $\mathrm{supp}\, f = \{x\in V : f(x)\neq 0\}$.

If $|\mathrm{supp}\, f| \le \frac{|V|}{2}$, then $B_f \ge h(G)\, \|f\|^2$.

Proof Since $|\mathrm{supp}\, f| \le \frac{|V|}{2}$, we have $\beta_0 = 0$,

and $|L_i| \le \frac{|V|}{2}$ for $i=1,\dots,r$. In particular,

$$|\delta(L_i)| \ge h(G)\, |L_i|.$$

This together with Step 1 gives

$$\mathcal{B}_f \geq h(G) \sum_{i=1}^{r} |L_i| \, (\beta_i^2 - \beta_{i-1}^2)$$

$$= h(G) \Big[|L_r| \beta_r^2 + (|L_{r-1}| - |L_r|) \beta_{r-1}^2 + \dots + (|L_1| - |L_2|) \beta_1^2 \Big]$$

$$= h(G) \Big[|L_r| \beta_r^2 + \sum_{i=1}^{r-1} |L_i \setminus L_{i+1}| \, \beta_i^2 \Big]$$

$$= h(G) \, \|f\|^2 .$$

because $L_i \setminus L_{i+1}$ is the level curve of β_i.

<u>Step 4</u> Putting things together.

Let $g \in \mathbb{R}^V$ be an eigenvector of Δ of eigenvalue $k - \lambda_2$.

Set $V^+ = \{ x \in V : g(x) > 0 \}$

$f = \max\{ g, 0 \}$.

We may assume $|V^+| \leq \frac{|V|}{2}$ (change g by $-g$ if necessary)

Also $|V^+| \neq 0$ because $\sum_{x \in V} g(x) = 0$.

For $x \in V^+$ we have

$$\begin{aligned}(\Delta f)(x) &= kf(x) - \sum_{y \in V} A_{xy} f(y) \\ &= kg(x) - \sum_{y \in V^+} A_{xy}\, g(y) \\ &\le kg(x) - \sum_{y \in V} A_{xy}\, g(y) \quad \text{because } g(y) \le 0 \text{ for } y \in V \setminus V^+. \\ &= (\Delta g)(x) = (k - \lambda_2)\, g(x) \\ &= (k - \lambda_2)\, f(x)\end{aligned}$$

Hence,

$$\begin{aligned}\| df \|^2 = \langle \Delta f, f \rangle &= \sum_{x \in V} \Delta f(x)\, f(x) \\ &\le (k - \lambda_2) \sum_{x \in V} f(x)^2 \\ &\le (k - \lambda_2)\, \| f \|^2.\end{aligned}$$

Step 2 and Step 3 imply

$$\begin{aligned}h(G)\, \| f \|^2 \le B_f &\le \sqrt{2k}\, \| df \|\ \| f \| \\ &\le \sqrt{2k(k - \lambda_2)}\ \| f \|^2.\end{aligned}$$

$\boxtimes$

§3 The bound of Alon and Boppana

Have seen: Large spectral gap connected to high expansion.

How big can the spectral gap be?

Thm ("Alon–Boppana bound")

There is a constant c such that for every k-regular graph on n vertices:

$$\lambda_2 \geq 2\sqrt{k-1}\left(1 - \frac{c}{\log^2 n}\right)$$

We will show a quantitative variation of this theorem:

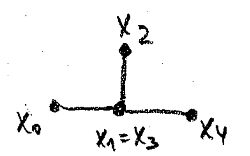
For any fixed $\varepsilon > 0$ a constant fraction of the n eigenvalues is at least $(2-\varepsilon)\sqrt{k-1}$.

Counting paths: $G = (V, E)$ simple graph

Def: A path of length r without backtracking is a sequence $(x_0, x_1, \dots, x_r)$ with:

- $x_i \in V \quad (i = 0, \dots, r)$
- $\{x_i, x_{i+1}\} \in E \quad (i = 0, \dots, r-1)$
- $x_{i+1} \neq x_{i-1} \quad (i = 1, \dots, r-1)$

Back-tracking:



Def: $A_r \in \mathbb{R}^{V \times V}$

$(A_r)_{i,j}$ = number of paths $(x_0, x_1, \dots, x_r)$ of length r without backtracking with $x_0 = i$ and $x_r = j$

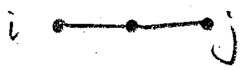
Note: $A_0 = I$, $A_1 = A$

Lemma: a) $A_1^2 = A_2 + k \cdot I$

b) For $r \geq 2$: $A_1 A_r = A_r A_1 = A_{r+1} + (k-1) A_{r-1}$

Proof: a) $(A_1^2)_{i,j} = (A^2)_{i,j} = \#$ of paths of length 2 between i and j

$\underline{i \neq j}$: No backtracking possible, $(A_1^2)_{i,j} = (A_2)_{i,j}$.



$\underline{i = j}$: G simple: $(A^2)_{i,i} = k$

$(A_2)_{i,i} = 0$ (only backtracking possible)

b) $\rightsquigarrow$ Exercise

$\square$

Now: Consider generating function of the A_r's, i.e., the formal power series with coefficients A_r.

Formal power series

R ring

$R[t]$ polynomial ring

$\rightsquigarrow$ finite sums $p_0 + p_1 t + p_2 t^2 + \dots + p_n t^m$
with $p_i \in R$

$R[[t]]$ ring of formal power series

$\rightsquigarrow$ (formal) infinite sums $\sum_{i=0}^{\infty} p_i t^i$, $p_i \in R$

- $\left(\sum_{i=0}^{\infty} p_i t^i\right) + \left(\sum_{i=0}^{\infty} q_i t^i\right) = \sum_{i=0}^{\infty} (p_i + q_i) t^i$

- $\left(\sum_{i=0}^{\infty} p_i t^i\right)\left(\sum_{i=0}^{\infty} q_i t^i\right) = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} p_j q_{i-j}\right) t^i$

Here, consider $\underline{\mathbb{C}^{V\times V}[[t]]}$.

<u>Lemma</u>: $$\left(\sum_{r=0}^{\infty} A_r t^r\right)\left(I - At + (k-1) I t^2\right) = I(1-t^2)$$

<u>Proof</u>: $\left(\sum_{r=0}^{\infty} A_r t^r\right)\left(I - At + (k-1) I t^2\right)$

$$= \sum_{r=0}^{\infty} A_r t^r - \sum_{r=1}^{\infty} A_{r-1} A t^r + (k-1) \sum_{r=2}^{\infty} A_{r-2} t^r$$

$$= \sum_{r=2}^{\infty} \underbrace{\left(A_r - A_{r-1}A + (k-1)A_{r-2}\right)}_{=0 \text{ for } r \geq 3} t^r + A_0 + \underbrace{A_1 t - A_0 A t}_{\substack{= At - IAt \\ = 0}}$$

$$= \left(A_2 - \underbrace{A_1^2}_{= A_2 + kI} + (k-1)I\right)t^2 + I = I - It^2$$

$\square$

To get a nicer formula:

$$T_m := \sum_{0 \leq r \leq \frac{m}{2}} A_{m-2r} = \begin{cases} A_0 + A_2 + \ldots + A_m & m \text{ even} \\ A_1 + A_3 + \ldots + A_m & m \text{ odd} \end{cases}$$

<u>Lemma:</u> $\left(\sum_{m=0}^{\infty} T_m t^m\right)\left(I - At + (k-1)It^2\right) = I$

<u>Proof:</u> (1) $\left(\sum_{r=0}^{\infty} I t^{2r}\right)\left(\sum_{r=0}^{\infty} A_r t^r\right) = \sum_{m=0}^{\infty} \Big(\underbrace{\sum_{0 \leq r \leq m/2} A_{m-2r}}_{= T_m}\Big) t^m$

(2) $\left(\sum_{r=0}^{\infty} I t^{2r}\right)\left(I - It^2\right) = \sum_{r=0}^{\infty} I t^{2r} - \sum_{r=0}^{\infty} I t^{2(r+1)} = I$

(1) & (2) & previous lemma:

$$\left(\sum_{m=0}^{\infty} T_m t^m\right)\left(I - At + (k-1)It^2\right) = \left(\sum_{r=0}^{\infty} I t^{2r}\right)\left(\sum_{r=0}^{\infty} A_r t^r\right)\left(I - At + (k-1)It^2\right)$$
$$= \left(\sum_{r=0}^{\infty} I t^{2r}\right)\left(I - It^2\right) = I$$

$\square$

Proposition: $T_m = (k-1)^{\frac{m}{2}} U_m\left(\frac{1}{2\sqrt{k-1}} A\right)$

where U_m are "Chebyshev polynomials of the second kind",

defined by recurrence relation

$$U_{m+1}(x) = 2x\,U_m(x) - U_{m-1}(x), \quad U_0(x) = I,$$
$$U_1(x) = 2x.$$

Proof: Using recurrence relation can show (as for A_r's):

$$\left(\sum_{m=0}^{\infty} (k-1)^{m/2} U_m\left(\frac{1}{2\sqrt{k-1}} X\right) t^m\right)\left(I - Xt + (k-1) I t^2\right) = I$$

So, both $\sum_{m=0}^{\infty} T_m t^m$ and $\sum_{m=0}^{\infty} (k-1)^{m/2} U_m\left(\frac{A}{2\sqrt{k-1}}\right) t^m$

are inverses of $(I - At + (k-1) I t^2)$,

and hence the same element in $\mathbb{C}^{V \times V}[[t]]$ $\square$

Thm: ("Trace formula")

$$\sum_{i \in V} \sum_{0 \le r \le \frac{m}{2}} (A_{m-2r})_{ii} = (k-1)^{m/2} \sum_{j=1}^{n} U_m\left(\frac{\lambda_j}{2\sqrt{k-1}}\right)$$

Proof: Consider trace of T_m:

$$\mathrm{Tr}(T_m) = (k-1)^{m/2}\, \mathrm{Tr}\left(U_m\left(\frac{1}{2\sqrt{k-1}} A\right)\right) = (k-1)^{m/2} \sum_{j=1}^{n} U_m\left(\frac{\lambda_j}{2\sqrt{k-1}}\right)$$

⌜ M matrix with eigenvalues $\mu_1, \dots, \mu_n$: $\mathrm{Tr}(\alpha \cdot M^r) = \alpha \cdot \mathrm{Tr}(M^r) = \sum_{i=1}^{n} \alpha \cdot \mu_i^r$

So: For polynomial p: $\mathrm{Tr}(p(M)) = \sum_{i=1}^{n} p(\mu_i)$ ⌟

This tells us that $\sum_{j=1}^{n} U_m\left(\frac{\lambda_j}{2\sqrt{k-1}}\right)$ is non-negative.

Now, use (without proof) the following property of U_m's:

<u>Proposition:</u>

For $\varepsilon > 0$ and $L \geq 2$ there exists a constant $C = C(\varepsilon, L) > 0$ with:

For any probability measure ν on $[-L, L]$ such that

$\int_{-L}^{L} U_m\left(\frac{x}{2}\right) d\nu(x) \geq 0$ for every $m \in \mathbb{N}$

we have $\quad \nu([2-\varepsilon, L]) \geq C$.

<u>Thm:</u>

For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, k) > 0$ with:

For every connected, finite, simple k-regular graph G on n vertices at least $C \cdot n$ eigenvalues of the adjacency matrix of G lie in $[(2-\varepsilon)\cdot\sqrt{k-1}, k]$.

<u>Proof:</u> Let $L = \frac{k}{\sqrt{k-1}} \geq 2$ and $\nu = \frac{1}{n} \cdot \sum_{j=1}^{n} \delta_{\frac{\lambda_j}{\sqrt{k-1}}}$

Dirac measure δ_a:

$$\delta_a(M) = \begin{cases} 0 & a \notin M \\ 1 & a \in M \end{cases}$$

$\int_{-L}^{L} f(x)\, d\delta_a(x) = f(a)$ for any cts f on $[-L, L]$

- ν probability measure on $[-L, L]$:

$$\nu([-L, L]) = \frac{1}{n} \cdot \left| \left\{ j : \frac{\lambda_j}{\sqrt{k-1}} \in \left[-\frac{k}{\sqrt{k-1}}, \frac{k}{\sqrt{k-1}}\right] \right\} \right| = 1$$

- $$\int_{-L}^{L} U_m\left(\frac{x}{2}\right) d\nu(x) = \frac{1}{n} \cdot \sum_{j=1}^{n} \int_{-L}^{L} U_m\left(\frac{x}{2}\right) d\delta_{\frac{\lambda_j}{\sqrt{k-1}}}(x)$$
$$= \frac{1}{n} \sum_{j=1}^{n} U_m\left(\frac{\lambda_j}{2\sqrt{k-1}}\right) \geq 0 \text{ f.a. } m \in \mathbb{N}$$

So, there is a constant $C = C(\varepsilon, k) > 0$: $\nu([2-\varepsilon, L]) \geq C$.

$$\nu([2-\varepsilon, L]) = \frac{1}{n} \cdot \left| \left\{ j : 2-\varepsilon \leq \frac{\lambda_j}{\sqrt{k-1}} \leq L \right\} \right|$$
$$= \frac{1}{n} \left| \left\{ j : (2-\varepsilon)\sqrt{k-1} \leq \lambda_j \leq k \right\} \right|$$

$\square$

§4 Margulis' construction

Non trivial fact: Families of expanders exist.

- Pinsker (1973): $\Omega_{n,k}$ = probability space of all k-regular graphs with vertex set $\{1, \dots, n\}$.

  Choose $X_{n,k}$ from $\Omega_{n,k}$ uniformly at random. Then for $k \geq 3$ $\exists\, \varepsilon(k) > 0$ so that

  $$\Pr[h(X_{n,k}) > \varepsilon(k)] \to 1 \text{ for } n \to \infty.$$

- Margulis (1973): Explicit construction of a family of 8-regular expander graphs.
- Gabber-Galil (1981): simplified Margulis' construction and analysis
- Here: follow blog of James Lee (2012) on Gabber-Galil analysis of Margulis' construction; further refined by Luca Trevisan

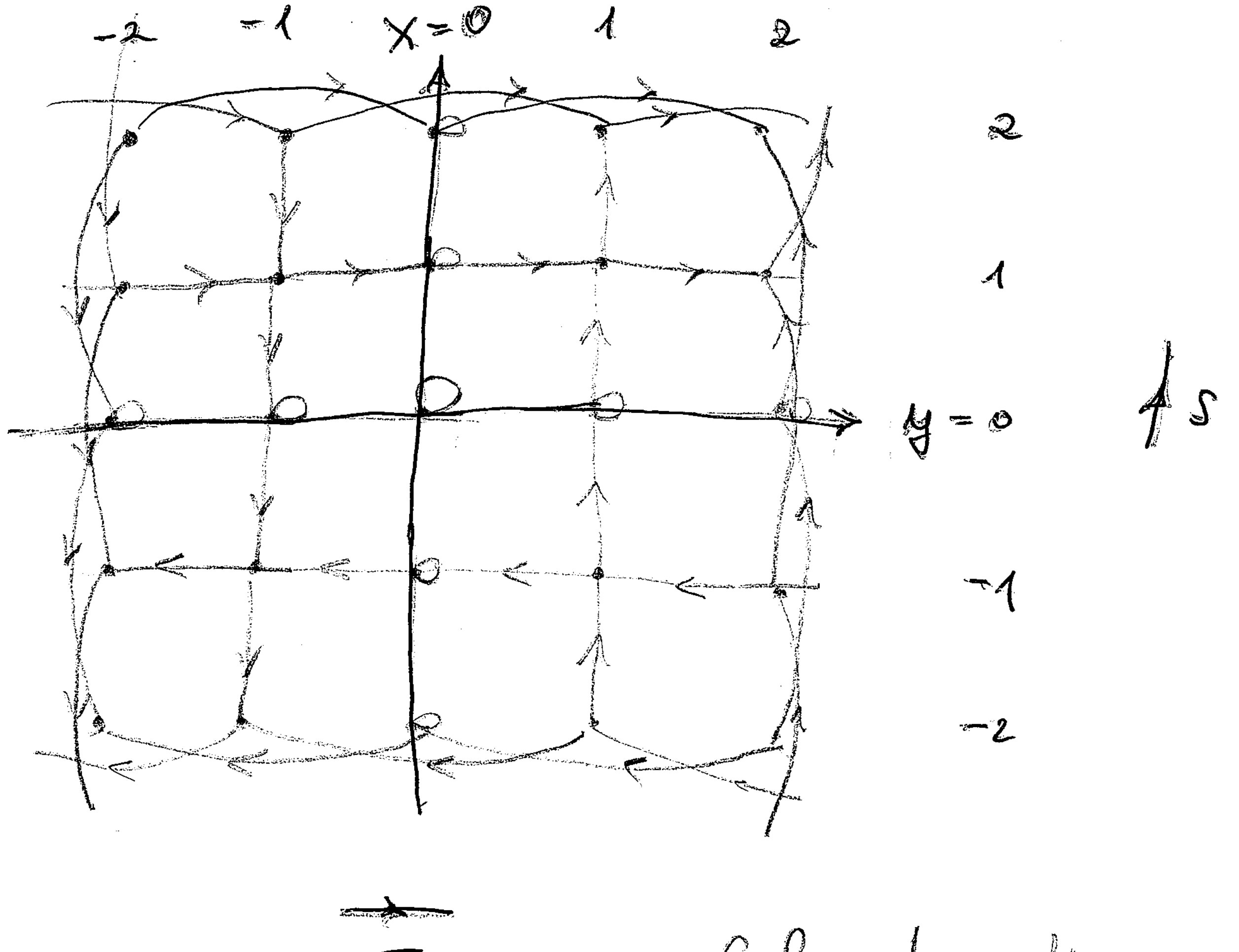
Def. Infinite graph G with vertex set $\mathbb{Z}^2$ and edges given by

$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$S\binom{x}{y} = (x, x+y)^T, \quad T\binom{x}{y} = (x+y, y)^T$$

where

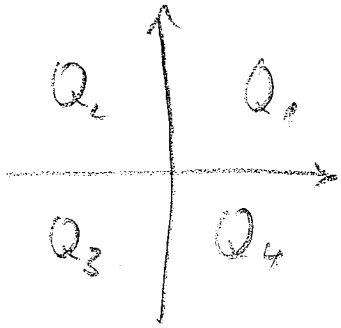
(x,y) is connected to $S(x,y)$, $T(x,y)$, $S^{-1}(x,y)$, $T^{-1}(x,y)$



0 has degree 4, loops count with multiplicity 2 or 4.

Lemma For every finite subset $A \subseteq \mathbb{Z}^2 \setminus \{0\}$ we have $$\frac{|\delta(A)|}{|A|} \geq \frac{2}{5}.$$

Proof First consider A not intersecting the coordinate axes. Let $Q_1, \dots, Q_4$ quadrant of $\mathbb{Z}^2$ and set $A_i = A \cap Q_i$



Note

- $S(A_1), T(A_1) \subseteq Q_1$: ✓
- $S(A_1) \cap T(A_1) = \emptyset$:

  For $(x,y) \in A_1$, $(x',y') \in A_1$ with $(x, x+y) = (x'+y', y')$ we would have

  $$x'+y'+y = y' \iff x' = -y \quad \lightning$$

- Similarly, $S^{-1}(A_2), T^{-1}(A_2) \subseteq Q_2$, $S^{-1}(A_2) \cap T^{-1}(A_2) = \emptyset$

  $S(A_3), T(A_3) \subseteq Q_3$, $S(A_3) \cap T(A_3) = \emptyset$

  $S^{-1}(A_4), T^{-1}(A_4) \subseteq Q_4$, $S^{-1}(A_4) \cap T^{-1}(A_4) = \emptyset$

$A_0 = \{(x,y) \in A : x = 0 \text{ or } y = 0\}$

Then $A = A_0 \cup \dots \cup A_4$.

Claim 1) $|$ edges connecting $A \setminus A^0$ to $\overline{A}\,| \geq |A| - |A_0|$

2) $|$ edges connecting A_0 to $\overline{A}\,| \geq 5|A_0| - 3|A|$

Hence $|\delta(A)| \geq |A| - |A_0| + \frac{1}{5}(5|A_0| - 3|A|)$

$= \frac{2}{5}|A|$

Proof

of 1) At least $|A_i|$ of the edges leaving A_i land in Q_i but not in A_i because if for instance $i = 1$ then $S(A_1), T(A_1) \subseteq Q_1$ and $S(A_1) \cap T(A_1) = \emptyset$.

of 2) All nonloop edges having an endpoint in A_0 have other endpoint outside of A_0. Some of those edges could land in $A \setminus A_0$. From $A \setminus A_0$ at most $4|A \setminus A_0|$ edges emanate, at least $|A| - |A_0|$ of them land in $\overline{A}$ by claim 1). So there are at most $3|A \setminus A_0|$ edges between A_0 and $A \setminus A_0$.

So

$$|\text{edges connecting } A_0 \text{ to } \bar{A}| \geq 2|A_0| - 3|A \setminus A_0|$$
$$= 5|A_0| - 3|A|.$$

$\boxtimes$

<u>So</u>: G is a good expander, but it's an infinite graph.

<u>Def.</u> $G_n = (V_n, E_n)$ with

$$V_n = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$$

and (x, y) is connected to

$$(x, y \pm 1), (x \pm 1, y), (x, x \pm y), (x \pm y, y).$$

G_n are called <u>Margulis expander</u>.

<u>Thm.</u> $\exists\, \varepsilon > 0$ so that for all n, the second smallest eigenvalue μ_2 of $\Delta(G_n)$ is at least ε.

Proof (sketch)

Consider 2^{nd} smallest eigenvalue of Laplacian of G_n

$$\mu_2(G_n) = \min_{\substack{f \in \mathbb{R}^{V_n} \setminus \{0\} \\ (f,e)=0}} \frac{(\Delta f, f)}{(f,f)}$$

where $(\Delta f, f) = \sum_{uv \in E_n} (f(u) - f(v))^2$.

Consider continuous graph C_n with vertex set $[0,n) \times [0,n)$ and vertex (x,y) is adjacent to $S(x,y)$, $S^{-1}(x,y)$, $T(x,y)$, $T^{-1}(x,y)$ where one computes mod n.

Define complex Hilbert space

$$L^2([0,n]^2) = \{ f : [0,n)^2 \to \mathbb{C} : \int |f(z)|^2 \, dz < \infty \}.$$

and

$$\mu_2(C_n) = \inf_{\substack{f \in L^2([0,n]^2) \setminus \{0\} \\ (f,e)=0}} \frac{\int |f(x,y) - f(S(x,y))|^2 + |f(x,y) - f(T(x,y))|^2 \, dxdy}{\|f\|_{L^2}}.$$

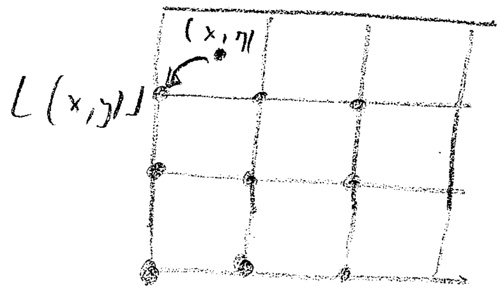
<u>Claim</u> $\exists\, \varepsilon' > 0 : \mu_2(G_n) \geq \varepsilon' \mu_2(C_n)$.

<u>Because</u> Let $f \in \mathbb{R}^{V_n}$ be a minimiser for $\mu_2(G_n)$

Extend f to $\tilde{f} : [0,n) \to \mathbb{C}$ as follows.

For $(x,y) \in [0,n)$ write $\lfloor (x,y) \rfloor = (\lfloor x \rfloor, \lfloor y \rfloor)$.

Then $\tilde{f}(x,y) = f(\lfloor x \rfloor, \lfloor y \rfloor)$.



<u>Clearly</u>

$$\int_{[0,n)^2} |\tilde{f}(x,y)|^2 \, dx\, dy = \sum_{\substack{(x,y) \\ \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}}} |f(x,y)|^2$$

and $(\tilde{f}, e) = (f, e) = 0$.

For every $z \in [0,n)^2$ we have

$$\lfloor S(z) \rfloor \in \{ S(\lfloor z \rfloor),\ S(\lfloor z \rfloor) + (0,1) \}$$
$$\lfloor T(z) \rfloor \in \{ T(\lfloor z \rfloor),\ T(\lfloor z \rfloor) + (1,0) \}.$$

Then

$$\int |\tilde{f}(x,y) - \tilde{f}(S(x,y))|^2 + |\tilde{f}(x,y) - \tilde{f}(T(x,y))|^2 \, dx\, dy$$

$$= \sum_{c=(a,b)\in(\mathbb{Z}/n\mathbb{Z})^2} \int_{\substack{[a,a+1)\\ \times[b,b+1)}} |\tilde{f}(z) - \tilde{f}(S(z))|^2 + |\tilde{f}(z) - \tilde{f}(T(z))|^2 \, dz$$

$$= \frac{1}{2} \sum_{c\in(\mathbb{Z}/n\mathbb{Z})^2} |f(c) - f(S(c))|^2 + |f(c) - f(S(c)+(0,1))|^2$$
$$+ |f(c) - f(T(c))|^2 + |f(c) - f(T(c)+(1,0))|^2$$

$= \ldots$ some more steps. $\boxtimes$

<u>Claim</u> $\mu_2(C_n) = \mu_2(G)$.

<u>Because</u> Use Fourier transform

$$\hat{\ } : L^2([0,n)) \longrightarrow \ell^2(\mathbb{Z}^2)$$

$$\{(u_n)_{n\in\mathbb{Z}^2} : \sum_{n\in\mathbb{Z}^2} |u_n|^2 < \infty\}$$

which is an isometry between $L^2([0,n))$ and $\ell^2(\mathbb{Z}^2)$.

Characters $\chi_{a,b}(x,y) = \frac{1}{n} e^{2\pi i (ax+by)}$.

<u>Claim</u> $\mu_2(G) \geq \frac{h(G)^2}{2 \cdot 8} \geq \frac{4}{25 \cdot 2 \cdot 8} = \frac{1}{100}$

Cheeger's inequality (proof also applies to G)

Lemma

$\boxtimes$

* * *