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Methods and problems in discrete mathematics

Wintersemester 2019/20

— Exercise Sheet 12 —

Exercise 12.1 For two non-zero vectors $x, y \in \mathbb{R}^r$ show:

(a) $\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle^2 \geq 1 - \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2$,

(b) $\|x\| \cdot \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq 2\|x - y\|$.

Exercise 12.2 Let $G = (V, E)$ be a k-regular graph without loops and let $V = U_1 \dot\cup U_2 \dot\cup \ldots \dot\cup U_r$ be a partition.

(a) Let f be a linear combination of the characteristic vectors of the U_i. Show:

$$\frac{\langle \Delta f, f \rangle}{\langle f, f \rangle} \leq 2 \cdot \max_{i=1\ldots,r} \phi_G(U_i).$$

(b) Show that $\frac{k-\lambda_r}{2} \leq k \cdot \rho_G(r)$ for the r-th largest eigenvalue λ_r of the adjacency matrix of G.

Exercise 12.3 Consider the random clustering algorithm described in the lecture. Show that the probability that a partition is formed after N rounds is at least

$$1 - n \left(\frac{n-1}{n} \right)^N,$$

where n is the number of points to be clustered.

Exercise 12.4 Describe the algorithm behind the proof of the higher order Cheeger inequality.

“Hand-in”: Until Thursday **January 23**, 10 am, using the form on the course homepage.