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Methods and problems in discrete mathematics

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— Tasks and Questions —

Die folgende Liste von Aufgaben soll die Vorbereitung auf eine schriftliche bzw. mündliche Prüfung über die Vorlesung Methods and problems in discrete mathematics (WS 2019/20) erleichtern. Die Aufgaben sind natürlich nur eine Auswahl von möglichen Aufgaben, sie sollten aber einen großen Teil des Vorlesungsstoffes abdecken. Die Aufgaben variieren in ihrem Schwierigkeitsgrad. Zu einer guten Vorbereitung raten wir zuerst jeweils ein Kapitel des Skriptes intensiv zu lesen und anschließend, die Aufgaben zu bearbeiten, ohne in das Skript zu schauen.

Chapter I — Universally optimal distribution of points
State the main theorem about universal optimal configurations on the unit sphere and provide all necessary definitions. Work out an example for which the main theorem applies. Give the statement of the linear programming bound together with its proof. State the spherical design strength test and provide all necessary definitions. Define the positivity property of the orthogonal polynomials P^n_k. What are orthogonal polynomials? Explain the statements about the 3-term recurrence relations and the interlacing of roots (make a picture). What is Hermite interpolation? How is Hermite interpolation used in the proof of the main theorem? Give the statement of the Gauss-Jacobi quadrature and prove it. What is a conductive polynomial? Show the multiplicativity of conductive polynomials. Explain how the spherical t-design property is used in proving the main theorem.

Chapter II — Harmonic analysis of (finite) groups
What is a G-invariant positive definite kernel? What is a unitary representation? How can one define a unitary representation using a G-invariant positive definite kernel? What are the central properties of the discrete Fourier transform? State Maschke's theorem and prove it. State Schur's lemma and prove it. Show the orthogonality relation. State the Peter-Weyl theorem. State Bochner's theorem and explain how it follows from the Peter-Weyl theorem. Compare the central properties of the DFT with the Fourier analysis of non commutative groups. Explain the relation between Bochner's theorem and Schoenberg's theorem What is an harmonic polynomial? What are spherical harmonics? Give examples. What is a zonal spherical function? How are they used to show that a representation is irreducible? State the Peter-Weyl theorem for the unit sphere. Explain the addition formula and relate it to the positivity property. Prove the spherical design strength test.

Chapter III — Expander graphs
Show that the multiplicity of the largest eigenvalue equals the number of connected components. What is a family of expanders? Explain why the cycle graphs are not a family of expanders. Why does large spectral gap imply high expansion? Make the statement precise and give mathematical arguments for it. Sketch the proof of Cheeger's inequality. State the bound of Alon and Boppana. What is the relation between walks and powers of the adjacency matrix? State the trace formula. Explain the construction of Margulis. What are the main steps in the proof that the Margulis construction gives a family of expander graphs (give the three relevant graphs and explain their properties)?

Chapter IV — Low distortion Euclidean embeddings
What is the minimal Euclidean distortion of a finite metric space? Derive a semidefinite program to compute the minimal Euclidean distortion of a finite metric space. What is its dual (consider the Corollary)? Which information can be derived from it? Show that $c_2(r\text{−cube}) = \sqrt{r}$. What is a strongly regular graph? Compute its eigenvalues. Sketch the proof that the least distortion embedding of an expander is $\log n$.

Chapter V — Higher order Cheeger inequalities
State the higher order Cheeger inequality and provide all necessary definitions. Describe the randomized partitioning algorithm and prove the bound on the probability that two points land in different clusters. What is the spectral embedding of a graph? How is it connected to the eigenvalues of the adjacency matrix? Sketch the proof of the higher order Cheeger inequality and describe the algorithm behind the proof.

Chapter VI — Quasirandomness in additive groups and hypergraphs
Show that a graph G satisfying $\|G-p\|_{\square} \leq \epsilon$ has triangle homomorphism density between $p^3-3\epsilon$ and $p^3+3\epsilon$. Obtain an upper bound for the second largest eigenvalue of the adjacency matrix of a graph G of edge density p satisfying $t(C_4, G) \leq p^4+\epsilon$. For a set $A \subset G$ (G an additive group), explain the relationship between the number of solutions $(x, y, z, w) \in A^4$ of the equation $x+y=z+w$ and the Fourier coefficients of A. Prove the identity $\|f\|_{U^2(G)} = \|\hat{f}\|_{\ell^4(\hat{G})}$ for a real-valued function f. Show that a set $A \subseteq G$ is linear uniform if and only if its Cayley graph Γ_A is quasirandom.