Quasirandomness in additive groups and hypergraphs

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Plan of the lectures

Style: slides (available on the website) and blackboard.

Contents:

- A few words on quasirandomness
- Quasirandom graphs
- Uniformity and quasirandomness in additive groups
- Quasirandomness in hypergraphs
- Comparing quasirandomness in additive groups and in hypergraphs

Mathematical style: An analytical view on discrete mathematics.

A few words on quasirandomness

Quasirandom objects

Intuitively, quasirandom objects are those that behave like a random object of the same type.

Main idea: We first identify the most important characteristics of a random mathematical object, and then define a quasirandom object of this type as one which shares these same characteristics.

This characteristic is usually related to the lack of correlation between distinct sub-parts of the object considered, which gives it strong uniformity properties.

Quasirandom objects

Motivation: Random objects are usually much easier to analyze, as their properties are highly concentrated around their mean (which is simple to compute).

In general, there are several decomposition theorems which allow us to decompose *any* object into a highly structured component and a quasirandom component (with possibly a small error term).

This is known in combinatorics as the *dichotomy between structure and randomness*, and is an essential ingredient to prove several very general results.

Asymptotic equivalence

We will show many natural "quasirandom properties" are roughly equivalent, in the following sense:

Definition: We say a property $P_1 = P_1(c_1)$ is asymptotically equivalent to a property $P_2 = P_2(c_2)$ if:

- $\forall c_1 > 0\ \exists c_2 > 0 :\ P_2(c_2) \Rightarrow P_1(c_1)$
- $\forall c_2 > 0\ \exists c_1 > 0 :\ P_1(c_1) \Rightarrow P_2(c_2)$

Example: Consider $P_1 : (x-2)^2 \leq c_1$ and $P_2 : |x^2 - 4| \leq c_2$

- P_1 and P_2 are asymptotically equivalent for $x \in \mathbb{R}_{\geq 0}$
- P_1 and P_2 are *not* asymptotically equivalent for $x \in \mathbb{R}$

General notation

We write $x = a \pm b$ to denote that $a - b \leq x \leq a + b$.

We use the expected value to denote average over a finite set, i.e.

$$\mathbb{E}_{x \in X}[\, f(x)\,] := \frac{1}{|X|} \sum_{x \in X} f(x)$$

We usually identify a set with its indicator function: given a set A, we write

$$A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Quasirandom graphs

Introduction

First and most well-known quasirandomness concept in combinatorics, which formally introduced the notion of quasirandom objects and motivated its study in other settings.

It was first introduced by Chung, Graham and Wilson in 1989, when they showed that several different-looking properties usually satisfied by random graphs are actually all roughly equivalent.

Intuition: A graph is quasirandom if its edge distribution resembles the one of a truly random graph with the same edge density.

Random graphs

The basic model of random graphs with edge density p is the *Erdös-Rényi random graph* $G(n, p)$:

> A graph on n vertices where every pair of vertices has probability p of being an edge, all choices independent.

Important property: The edges are distributed very uniformly.

Definition: Given a graph G and two sets $A, B \subseteq V(G)$, we define the *cut between A and B in G* as

$$E_G(A, B) := \{(x, y) \in A \times B : \; xy \in E(G)\}$$

Random graphs

For the Erdös-Rényi random graph $G = G(n, p)$:

> Each one of the $|A||B|$ pairs $(x, y) \in A \times B$ has probability p of being in $G(n, p)$, so the expected size of $E_G(A, B)$ is
>
> $$\mathbb{E}\left[\sum_{x\in A}\sum_{y\in B} G(x,y)\right] = \sum_{x\in A}\sum_{y\in B} \mathbb{P}\left(G(x,y)=1\right) = p|A||B|$$
>
> The actual number of edges in this cut is highly concentrated around their mean $p|A||B|$, with error $o(n^2)$.

If a graph satisfies this uniform distribution of edges over all cuts, we say it is *quasirandom*.

Notation: For a graph G, we denote its number of edges by $|G|$ and its number of vertices by $v(G)$.

Theorem (Chung, Graham, Wilson):
Let G be a graph with n vertices and edge density p. Then the following statements are asymptotically equivalent:

(i) For any two subsets $A, B \subseteq V(G)$, the size of the cut $E_G(A, B)$ differs from $p|A||B|$ by at most $c_1 n^2$

(ii) The number of labelled copies of any given graph F in G differs from $p^{|F|} n^{v(F)}$ by at most $c_2 |F| n^{v(F)}$

(iii) The number of labelled 4-cycles in G is at most $(p^4 + c_3) n^4$

(iv) The largest eigenvalue of the adjacency matrix of G is $(p \pm c_4) n$, and all other eigenvalues are at most $c_4 n$ in absolute value

Remarks

These properties are all satisfied by the Erdös-Rényi random graph with high probability:

- As discussed, the size of the cut $E_G(A, B)$ is highly concentrated around its mean $p|A||B|$.
- There are $n^{v(F)}$ ways of choosing in the vertices for a copy of F in $G(n, p)$, and each of the $|F|$ edges have probability p of being in $G(n, p)$. The expected number of copies of F is then $p^{|F|}n^{v(F)}$, and it will be very close to the mean with high probability.
- Item (iv) is a well-known property of random symmetric matrices, and for graphs it is related to the spectral properties of edge expansion we have seen earlier in the course.

Cut norm

Intuition: The cut norm measures the discrepancy of the edge distribution of the graph, that is, how far from uniform the edge distribution is.

For graphs: It measures how much the size of a cut $E_G(A, B)$ can deviate from its "expected" value $p|A||B|$, over all sets $A, B \subseteq V$:

$$\frac{1}{|V|^2} \max_{A,B \subseteq V} ||E_G(A, B)| - p|A||B||$$

It is convenient to first "balance" the graph by subtracting its density, considering instead of the graph G its balanced function $f_G := G - p$.

Cut norm

Definition: Given a function $f : V \times V \to \mathbb{R}$, its cut norm is

$$\|f\|_\square = \frac{1}{|V|^2} \max_{A,B \subseteq V} \left| \sum_{x \in A} \sum_{y \in B} f(x,y) \right|$$
$$= \max_{A,B \subseteq V} \left| \mathbb{E}_{x,y \in V} \left[f(x,y) 1_A(x) 1_B(y) \right] \right|$$

Remark: For any functions $u, v : V \to [0,1]$ we have that

$$\left| \mathbb{E}_{x,y \in V} \left[f(x,y) u(x) v(y) \right] \right| \le \|f\|_\square$$

With this notation, item (i) becomes $\|G - p\|_\square \le c_1$.

Homomorphism densities

Homomorphism densities are a convenient way of counting copies of small graphs inside large graphs.

Definition: The homomorphism density of a graph F in a graph G, denoted $t(F, G)$, is the probability that a randomly chosen map $\phi : V(F) \to V(G)$ preserves edges:

$$\begin{aligned} t(F, G) &= \mathbb{P}_{x_1, \ldots, x_{v(F)} \in V(G)} \left(x_i x_j \in E(G) \text{ whenever } ij \in E(F) \right) \\ &= \mathbb{E}_{x_1, \ldots, x_{v(F)} \in V(G)} \left[\prod_{ij \in E(F)} G(x_i, x_j) \right] \end{aligned}$$

With this notation, items (ii) and (iii) become $t(F, G) = p^{|F|} \pm c_2 |F|$ and $t(C_4, G) \leq p^4 + c_3$, respectively.

Quasirandom graphs (again)

Theorem (Chung, Graham, Wilson):
Let G be a graph with n vertices and edge density p. Then the following statements are asymptotically equivalent:

(i) *G has low discrepancy*: $\|G - p\|_\square \leq c_1$

(ii) *G correctly counts all graphs*: $t(F, G) = p^{|F|} \pm c_2|F|$ for all F

(iii) *G has few 4-cycles*: $t(C_4, G) \leq p^4 + c_3$

(iv) *Only the first eigenvalue matters*: $\lambda_1 = (p \pm c_4)n$, $|\lambda_2| \leq c_4 n$

Uniformity and quasirandomness in additive groups

Additive groups

Another fruitful setting for studying quasirandomness is subsets of additive groups, which are the main object of study in additive combinatorics.

Definition: An *additive group* is any Abelian group G with group operation $+$.

Remark: All additive groups we consider here will be *finite*.
While perhaps the most important additive group is $\mathbb{Z}$, it can be studied by analyzing $\mathbb{Z}_N$ for N large enough.

Intuition: Quasirandom sets are those which have many of the same properties as a randomly chosen set of the same density.

An interesting way of measuring this is using the *Fourier transform*.

Fourier analysis on additive groups

Quick recap:

- The characters of an additive group G are the group homomorphisms from G to $\mathbb{C}^\times$ (the complex multiplicative group).
- They form a group (with pointwise multiplication) called the *dual group* of G, and denoted by $\widehat{G}$.
- They also form an orthonormal basis of $\mathbb{C}^G$, with inner product given by $\langle f, g \rangle_{L^2(G)} := \mathbb{E}_{x \in G}[f(x)\overline{g(x)}]$.

Definition: Given a function $f : G \to \mathbb{C}$, we define its *Fourier transform* $\widehat{f} : \widehat{G} \to \mathbb{C}$ by

$$\widehat{f}(\gamma) := \langle f, \gamma \rangle_{L^2(G)} = \mathbb{E}_{x \in G}[f(x)\overline{\gamma(x)}]$$

Example: If $G = \mathbb{Z}_n$, we have the usual discrete Fourier transform. In this case:

- The characters are $\chi_r : x \mapsto e^{2\pi irx/n}$, for $r \in \mathbb{Z}_n$
- The Fourier transform is given by $\widehat{f}(\chi_r) = \mathbb{E}_{x\in\mathbb{Z}_n}[f(x)e^{-2\pi irx/n}]$

Remark: Together with the *fundamental theorem of finite Abelian groups*, this example gives an explicit formula for the characters and the Fourier transform in any additive group G. However, we will not need such an explicit description here.

Fourier analysis on additive groups

While G and $\widehat{G}$ are isomorphic, it is convenient to use different measures on them:

- For G we use the normalized measure $\mathbb{E}_{x \in G}$, and denote the associated Euclidean space $\mathbb{C}^G$ by $L^2(G)$.
- For $\widehat{G}$ we use the counting measure $\sum_{\gamma \in \widehat{G}}$, and denote the associated Euclidean space $\mathbb{C}^{\widehat{G}}$ by $\ell^2(\widehat{G})$.

This way, the Fourier transform is an *isometry* from $L^2(G)$ to $\ell^2(\widehat{G})$.

Since the characters form an orthonormal basis of $L^2(G)$, we have

$$f(x) = \sum_{\gamma \in \widehat{G}} \langle f, \gamma \rangle_{L^2(G)} \gamma(x) = \sum_{\gamma \in \widehat{G}} \widehat{f}(\gamma) \gamma(x)$$

Connection to quasirandomness:

The Fourier transform of a function f evaluated at a character γ gives how much f correlates with γ.

As the characters of G encode the additive structure of the group, correlation with a (non-trivial) character is a good measure of how much additive structure a given function or set has.

Definition: A function $f : G \to \mathbb{R}$ is said to be *linear ϵ-uniform* if $|\widehat{f}(\gamma)| \leq \epsilon$ for all $\gamma \in \widehat{G} \setminus \{1\}$. A set $A \subseteq G$ is linear ϵ-uniform if its indicator function is.

Theorem (Chung, Graham):
Let G be an additive group of order n and let $A \subseteq G$ be a set of size $|A| = \alpha n$. Then the following are asymptotically equivalent:

(i) *Fourier coefficients:* $|\hat{A}(\gamma)| \leq c_1$ for all non-trivial characters γ

(ii) *Additive quadruples:* There are at most $(\alpha^4 + c_2)n^3$ solutions in A of the equation $x + y = z + w$

(iii) *Strong translation:* For all sets $B \subseteq G$, all but at most $c_3 n$ elements $x \in G$ satisfy $|A \cap (B + x)| = \alpha|B| \pm c_3 n$

(iv) *Weak translation:* All but at most $c_4 n$ elements $x \in G$ satisfy $|A \cap (A + x)| = \alpha^2 n \pm c_4 n$

The U^2 uniformity norm

For the proof of the theorem it is useful to introduce a new norm which measures the uniformity of a function:

Definition: Given a real function $f : G \to \mathbb{R}$, we define its U^2 norm by

$$\| f \|_{U^2(G)}^4 := \mathbb{E}_{x,h_1,h_2 \in G} \left[f(x) f(x+h_1) f(x+h_2) f(x+h_1+h_2) \right]$$

Note: This is the weighted count of additive quadruples $x + y = z + w$.

Connection to the Fourier transform:
For all real functions $f : G \to \mathbb{R}$, we have $\| f \|_{U^2(G)} = \| \widehat{f} \|_{\ell^4(\widehat{G})}$.

The proof is a simple application of the *orthogonality relations of characters*: $\mathbb{E}_{x \in G} [\gamma(x)] = 1_{\{\gamma = 1\}}$, $\sum_{\gamma \in \widehat{G}} \gamma(x) = |G| 1_{\{x=0\}}$.

The U^2 uniformity norm

$$\| f \|_{U^2(G)} := \mathbb{E}_{x,h_1,h_2 \in G} \left[f(x)f(x+h_1)f(x+h_2)f(x+h_1+h_2) \right]^{1/4}$$

Analogously to Euclidean spaces, we can define an *inner product of second order* by

$$\langle f_1, f_2, f_3, f_4 \rangle_{U^2(G)} = \mathbb{E}_{x,h_1,h_2 \in G} \left[f_1(x)f_2(x+h_1)f_3(x+h_2)f_4(x+h_1+h_2) \right],$$

and with this inner product we have $\| f \|_{U^2(G)} = \langle f, f, f, f \rangle_{U^2(G)}^{1/4}$.

Lemma (Gowers-Cauchy-Schwarz inequality):

For any functions $f_1, f_2, f_3, f_4 : G \to \mathbb{R}$ we have

$$\langle f_1, f_2, f_3, f_4 \rangle_{U^2(G)} \leq \| f_1 \|_{U^2(G)} \| f_2 \|_{U^2(G)} \| f_3 \|_{U^2(G)} \| f_4 \|_{U^2(G)}$$

Uniform subsets of G (again)

Theorem (Chung, Graham):
Let G be an additive group of order n and let $A \subseteq G$ be a set of size $|A| = \alpha n$. Then the following are asymptotically equivalent:

(i) *Fourier coefficients:* $|\hat{A}(\gamma)| \leq c_1$ for all non-trivial characters γ

(ii) *Additive quadruples:* There are at most $(\alpha^4 + c_2)n^3$ solutions in A of the equation $x + y = z + w$, or $\|A\|_{U^2(G)}^4 \leq \alpha^4 + c_2$

(iii) *Strong translation:* For all sets $B \subseteq G$, all but at most $c_3 n$ elements $x \in G$ satisfy $|A \cap (B + x)| = \alpha|B| \pm c_3 n$

(iv) *Weak translation:* All but at most $c_4 n$ elements $x \in G$ satisfy $|A \cap (A + x)| = \alpha^2 n \pm c_4 n$

Definition: Given a subset $A \subseteq G$, we define its Cayley graph Γ_A by

$$V(\Gamma_A) = G, \quad E(\Gamma_A) = \{xy : x + y \in A\}$$

Remark: This is slightly different from the usual definition of a Cayley graph, but is more convenient for us.

Lemma: A set $A \subseteq G$ is linear uniform if and only if its Cayley graph Γ_A is quasirandom.

Counting linear patterns in A

We have seen that linear uniformity suffices for us to count additive quadruples in A:

> If A is linear ϵ-uniform, then it contains between $\alpha^4 n^3$ and $(\alpha^4 + \epsilon^2)n^3$ solutions to the equation $x + y = z + w$.

What other linear patterns can we count in A by knowing it is linear uniform?

Lemma: Let G be an additive group of odd order and suppose $A \subseteq G$ is linear ϵ-uniform. Then there are between $(\alpha^3 - \epsilon)n^2$ and $(\alpha^3 + \epsilon)n^2$ 3-term arithmetic progressions in A.

Higher degree uniformity and Gowers norms

For more complicated patterns, we need to define stronger norms which control a stronger form of quasirandomness:

Linear uniformity:

$$\|f\|_{U^2(G)}^4 := \mathbb{E}_{x,h_1,h_2\in G}\left[f(x)f(x+h_1)f(x+h_2)f(x+h_1+h_2)\right]$$

Counts the (weighted) number of additive quadruples, or squares.

Quadratic uniformity:

$$\begin{aligned}\|f\|_{U^3(G)}^8 := \mathbb{E}_{x,h_1,h_2,h_3\in G}[&f(x)f(x+h_1)f(x+h_2)f(x+h_1+h_2)\\ &\times f(x+h_3)f(x+h_1+h_3)f(x+h_2+h_3)f(x+h_1+h_2+h_3)]\end{aligned}$$

Counts the weighted number of parallelepipeds.

Higher degree uniformity and Gowers norms

In general, for every $k \geq 2$ we can define the **uniformity norm of degree** k by

$$\|f\|_{U^k(G)}^{2^k} := \mathbb{E}_{x,h_1,\ldots,h_k \in G}\left[\prod_{\omega \in \{0,1\}^k} f(x + \omega \cdot \mathbf{h})\right],$$

where $\omega \cdot \mathbf{h} := \omega_1 h_1 + \cdots + \omega_k h_k$.

Remark: If $A \subseteq G$ is a set, then

$$\|A\|_{U^k(G)}^{2^k} = \mathbb{P}_{x,h_1,\ldots,h_k \in G}\left(x + \{0,1\}^k \cdot (h_1, \ldots, h_k) \subseteq A\right)$$

is the proportion of parallelepipeds of dimension k contained in A. By Cauchy-Schwarz, this value is at least α^{2^k}.

Higher degree uniformity and Gowers norms

Quadratic uniformity suffices to count how many 4-term arithmetic progressions $(x, x+r, x+2r, x+3r)$ are contained in A:

$$\mathbb{E}_{x,r\in G}\left[A(x)A(x+r)A(x+2r)A(x+3r)\right] = \alpha^4 \pm 4\,\|\,A-\alpha\,\|_{U^3(G)}$$

In general, uniformity of degree k is sufficient to count $(k+2)$-term arithmetic progressions in A, as well as several other linear patterns said to have complexity at most k.

Application: Green and Tao used such a result to compute the asymptotic number of solutions to systems of linear equations inside the first n primes.

Linear equations in primes (Annals of Mathematics, 2010)

Quasirandomness in hypergraphs

Hypergraphs

Hypergraphs are the natural generalization of graphs where edges can contain more than two vertices.

Definition: For $k \geq 2$, a *k-uniform hypergraph* (or *k-graph*) H is given by a vertex set $V(H)$ and an edge set $E(H) \subseteq \binom{V(H)}{k}$ (i.e. the k-element subsets of $V(H)$). We usually identify H with its edge set $E(H)$, and denote the number of vertices of H by $v(H)$.

As with graphs, quasirandom hypergraphs are those whose edge distribution resembles the one of a truly random hypergraph of the same edge density.

Random hypergraphs

First of all, how do we "naturally" choose a random hypergraph?
And what would be a good measure of quasirandomness?

For graphs (2-uniform hypergraphs):

- Model of random graph $G(n,p)$: n vertices, each pair of vertices has probability p of being an edge
- For a two-variable function $f : V \times V \to \mathbb{R}$ we define the *cut norm*

$$\|f\|_\square = \max_{A,B \subseteq V} |\mathbb{E}_{x,y \in V}[f(x,y)A(x)B(y)]|$$

- A graph G with edge density p is ϵ-quasirandom if $\|G - p\|_\square \leq \epsilon$ (meaning the edges are uniformly distributed inside all cuts)
- If G is quasirandom, then it contains about $n^{v(F)}p^{|F|}$ copies of any graph F as a subgraph

Random hypergraphs

Let us then try to generalize these notions to higher hypergraphs:

For 3-uniform hypergraphs:

- Model of random hypergraphs: n vertices, each *triple* of vertices has probability p of being an edge
- For a three-variable function $f : V \times V \times V \to \mathbb{R}$ define

$$\|f\|_{\square_1^3} = \max_{A,B,C \subseteq V} |\mathbb{E}_{x,y,z \in V}[f(x,y,z)A(x)B(y)C(z)]|$$

- Let us say that a 3-uniform hypergraph H with edge density p is ϵ-*quasirandom* if $\|H - p\|_{\square_1^3} \leq \epsilon$
- Then the random hypergraph defined is very quasirandom w.h.p, and it contains about $n^{v(F)}p^{|F|}$ copies of any hypergraph F

Random hypergraphs

There is another natural way of choosing a random 3-uniform hypergraph, which is by making random choices at the *second level* instead of the third:

> First pick a random graph $G = G(n, 1/2)$, then let H be the hypergraph corresponding to the triangles in G.

This random hypergraph H will indeed be very quasirandom by our earlier definition, but the counting lemma does *not* hold!

Example: Let F be the 3-uniform hypergraph on four vertices with two edges. Then the number of copies of F we would expect to find in H is about $n^4/64$, while its true number is about $n^4/32$.

The problem: The hypergraph F considered has edges intersecting at *two vertices*, while the cut norm used measures correlation with functions of *one vertex* at a time.

If we consider only *linear hypergraphs*, i.e. those where any two edges share at most one vertex, then this issue does not happen.

In order to control the number of copies of all 3-graphs, we need to consider the following stronger norm to measure quasirandomness:

$$\| f \|_{\square_2^3} = \max_{A,B,C \subseteq V \times V} \left| \mathbb{E}_{x,y,z \in V} \left[f(x,y,z) A(x,y) B(x,z) C(y,z) \right] \right|$$

Random hypergraphs

In general, to choose a random k-graph H we can make random choices at any level $2 \leq j \leq k$, or any subset of them:

> **Randomness at level j:**
>
> Pick each j-set $f \in \binom{V(H)}{j}$ at random with probability p_j, and let $e \in \binom{V(H)}{k}$ be an edge iff all its j-subsets $f \in \binom{e}{j}$ were chosen.

Example: To choose a random 3-graph H on the vertex set V, pick:

- A random subset $G^{(2)} \subseteq \binom{V}{2}$ of all pairs of vertices (each being in $G^{(2)}$ with probability p_2);
- A random subset $G^{(3)} \subseteq \binom{V}{3}$ of all triples of vertices (each being in $G^{(3)}$ with probability p_3).

Then $\{x, y, z\} \in \binom{V}{3}$ is an edge of H iff $\{x, y, z\} \in G^{(3)}$ and each pair $\{x, y\}, \{x, z\}, \{y, z\}$ is in $G^{(2)}$.

Quasirandom hypergraphs

For each level of randomness in the choice of a random k-graph there is an associated notion of quasirandomness.

Intuition: For $1 \leq d \leq k-1$, we say a k-graph H is quasirandom of order $d+1$ if it does not correlate with any structure of order d.

Definition: Given a function $f : V^{[k]} \to \mathbb{R}$ and an integer $1 \leq d \leq k-1$, we define the (k,d)-cut norm of f by

$$\|f\|_{\square_d^k} := \max_{u_B : V^B \to [0,1],\ \forall B \in \binom{[k]}{d}} \left| \mathbb{E}_{\mathbf{x} \in V^{[k]}} \left[f(\mathbf{x}) \prod_{B \in \binom{[k]}{d}} u_B(\mathbf{x}_B) \right] \right|$$

Notation: Given a k-tuple $\mathbf{x} \in V^{[k]}$ and a subset $B \subseteq [k]$, we denote by $\mathbf{x}_B := (x_i : i \in B)$ the projection of $\mathbf{x}$ into its B-coordinates.

Quasirandom hypergraphs

$$\| f \|_{\square^k_d} := \max_{u_B : V^B \to [0,1],\, \forall B \in \binom{[k]}{d}} \left| \mathbb{E}_{\mathbf{x} \in V^{[k]}} \left[f(\mathbf{x}) \prod_{B \in \binom{[k]}{d}} u_B(\mathbf{x}_B) \right] \right|$$

We say that a k-graph H of edge density p is ϵ-quasirandom of order $d+1$ if $\| H - p \|_{\square^k_d} \leq \epsilon$.

For random hypergraphs:

If all levels of randomness involved in the choosing of a random hypergraph H are $\geq d$, then H will be quasirandom of order d w.h.p.

If there is a non-trivial level of randomness smaller than d, then H will *not* be quasirandom of order d.

Counting subhypergraphs

What kind of information can we obtain from quasirandomness?

Definition: Let H and F be two k-uniform hypergraphs. The homomorphism density of F in H is the probability that a randomly chosen map $\phi : V(F) \to V(H)$ preserves edges:

$$\begin{aligned} t(F,H) &= \mathbb{P}_{x_1,\ldots,x_{v(F)}\in V(H)}\left(\{x_i : i \in e\} \in E(H) \text{ for all } e \in E(F)\right) \\ &= \mathbb{E}_{\mathsf{x}\in V(H)^{V(F)}}\left[\prod_{e\in E(F)} H(\mathsf{x}_e)\right] \end{aligned}$$

Remark: This is essentially the (normalized) number of copies of F inside H.

As noted before, quasirandomness of order 2 suffices to control the number of linear hypergraphs. In general, quasirandomness of order $d+1$ suffices to control the number of *d-linear hypergraphs*:

Definition: Let $d < k$ be positive integers. We say that a k-graph H is *d-linear* if every pair of edges intersect in at most d vertices.
We denote the set of all d-linear k-graphs by $\mathcal{L}_d^{(k)}$.

Counting lemma for quasirandomness of order $d+1$:

For any k-uniform hypergraph H and any $0 \leq p \leq 1$, we have that

$$t(F, H) = p^{|F|} \pm |F| \cdot \| H - p \|_{\square_d^k} \qquad \forall F \in \mathcal{L}_d^{(k)}$$

Necessity of d-linearity

We can generalize our previous example to show the assumption of d-linearity is necessary for the counting lemma:

Example: For $2 \leq d \leq k-1$, let F be the connected k-graph on $2k-d$ vertices and two edges. Note that F is d-linear.

Choose a random subset $G^{(d)} \subseteq \binom{V}{d}$ of all d-tuples of vertices, each being in $G^{(d)}$ with probability 1/2, and let H be the random k-graph on V where $\{x_1, \ldots, x_k\} \in \binom{V}{k}$ is an edge of H iff all its d-element subsets are in $G^{(d)}$.

Then with high probability H will be $o(1)$-quasirandom of order d and have density $p = 2^{-\binom{k}{d}} + o(1)$, but

$$t(F, H) = 2^{-2\binom{k}{d}+1} + o(1) \;\approx\; 2p^{|F|}$$

Quasirandomness of order $d + 1$

Theorem (Towsner):
Let H be a k-uniform hypergraph with edge density p. Then the following statements are asymptotically equivalent:

(i) H is quasirandom of order $d+1$: $\quad \|H - p\|_{\square^k_d} \leq c_1$

(ii) H correctly counts all d-linear hypergraphs:

$$t(F, H) = p^{|F|} \pm c_2|F| \qquad \forall F \in \mathcal{L}_d^{(k)}$$

(iii) H correctly counts $\mathrm{M}_d^{(k)}$: $\quad t(\mathrm{M}_d^{(k)}, H) = p^{|\mathrm{M}_d^{(k)}|} \pm c_3$

Here $\mathrm{M}_d^{(k)}$ is a specific d-linear hypergraph on $k2^{\binom{k-1}{d}}$ vertices and $2^{\binom{k}{d}}$ edges, constructed to model the applications of Cauchy-Schwarz needed in the proof that (ii) implies (i).

Strong quasirandomness

In order to control the number of *every* subhypergraph F in a k-graph H, we need H to be quasirandom of order k.

We say that such hypergraphs are *strongly quasirandom*.

Example: The random k-graph chosen by picking each k-tuple of vertices to be an edge independently with probability p will be strongly quasirandom with high probability.

Kohayakawa, Rödl and Skokan showed that a k-graph being strongly quasirandom is asymptotically equivalent to it having the almost minimal number of copies of the *k-octahedron* $\textsc{Oct}^{(k)}$:

$$V = \{x_1^{(0)}, x_1^{(1)}, \ldots, x_k^{(0)}, x_k^{(1)}\}, \quad E = \left\{\{x_1^{(\omega_1)}, \ldots, x_k^{(\omega_k)}\} : \omega \in \{0,1\}^k\right\}$$

Strong quasirandomness

Theorem (Kohayakawa, Rödl, Skokan):
Let H be a k-uniform hypergraph with edge density p. Then the following statements are asymptotically equivalent:

(i) H is strongly quasirandom: $\|H - p\|_{\square^k_{k-1}} \leq c_1$

(ii) H correctly counts *all* hypergraphs:

$$t(F, H) = \alpha^{|F|} \pm c_2|F| \quad \text{for all k-hypergraphs } F$$

(iii) H has few octahedra: $t(\textsc{Oct}^{(k)}, H) \leq p^{2^k} + c_3$

Octahedral norms

Definition: Given $f : V^k \to \mathbb{R}$, we define its *octahedral norm* by

$$\|f\|_{\textsc{Oct}^k} := \mathbb{E}_{\mathsf{x}^{(0)},\mathsf{x}^{(1)} \in V^k} \left[\prod_{\omega \in \{0,1\}^k} f(\mathsf{x}^{(\omega)}) \right]^{1/2^k},$$

where we write $\mathsf{x}^{(\omega)} := (x_i^{(\omega_i)})_{i \in [k]}$.

Note that $\|f\|_{\textsc{Oct}^k}^{2^k} = t(\textsc{Oct}^{(k)}, f)$ is the weighted count of k-octahedra.

It has an associated *inner product of order k*, denoted $\langle \cdot \rangle_{\textsc{Oct}^k}$, which we define for functions $f_\omega : V^k \to \mathbb{R}$, $\omega \in \{0,1\}^k$, by

$$\langle (f_\omega)_{\omega \in \{0,1\}^k} \rangle_{\textsc{Oct}^k} := \mathbb{E}_{\mathsf{x}^{(0)},\mathsf{x}^{(1)} \in V^k} \left[\prod_{\omega \in \{0,1\}^k} f_\omega(\mathsf{x}^{(\omega)}) \right]$$

Octahedral norms

$$\langle (f_\omega)_{\omega\in\{0,1\}^k}\rangle_{\mathrm{OCT}^k} := \mathbb{E}_{\mathsf{x}^{(0)},\mathsf{x}^{(1)}\in V^k}\left[\prod_{\omega\in\{0,1\}^k} f_\omega(\mathsf{x}^{(\omega)})\right]$$

With this inner product, we have $\|f\|_{\mathrm{OCT}^k}^{2^k} = \langle f, f, \ldots, f\rangle_{\mathrm{OCT}^k}$

Lemma (Gowers-Cauchy-Schwarz inequality):

For any collection of functions $f_\omega : V^k \to \mathbb{R}$, $\omega \in \{0,1\}^k$, we have

$$\langle (f_\omega)_{\omega\in\{0,1\}^k}\rangle_{\mathrm{OCT}^k} \leq \prod_{\omega\in\{0,1\}^k} \|f_\omega\|_{\mathrm{OCT}^k}$$

Corollary: For all $f : V^k \to \mathbb{R}$ we have that $\|f\|_{\Box^k_{k-1}} \leq \|f\|_{\mathrm{OCT}^k}$.

Comparing quasirandomness in additive groups and in hypergraphs

Cayley hypergraphs

Goal: To understand how the notions of quasirandomness for additive groups and for hypergraphs relate to each other.

Definition: For a subset $A \subseteq G$ of an additive group G, define the *k-uniform Cayley hypergraph* $H^{(k)}A$ by:

$$V = G, \qquad E = \left\{ \{x_1, \ldots, x_k\} \in \binom{G}{k} \,:\, x_1 + \cdots + x_k \in A \right\}$$

Remark: More generally, we can define a “Cayley-like hypergraph” for any linear form $\psi : G^k \to G$, or any system of such linear forms.

The theory is very similar, but with a heavier notation.

We have already seen a strong connection between *linear* uniform sets and their associated Cayley graphs:

Lemma: A set $A \subseteq G$ is linear uniform if and only if its Cayley graph Γ_A is quasirandom.

Today we will generalize this result and obtain a similar connection between uniform sets of degree d and quasirandomness of order d of their Cayley hypergraphs.

We will also see that such a condition gives a stronger control on the count of subhypergraphs than what we have in general quasirandom hypergraphs.

Recap: Quasirandomness in hypergraphs

There are many different notions of quasirandomness associated with k-uniform hypergraphs.

Intuition: For $1 \leq d \leq k-1$, we say a k-graph H is quasirandom of order $d+1$ if it does not correlate with any structure of order d.

Definition: Given a function $f : V^{[k]} \to \mathbb{R}$ and an integer $1 \leq d \leq k-1$, we define the (k,d)-cut norm of f by

$$\| f \|_{\square^k_d} := \max_{u_B : V^B \to [0,1],\, \forall B \in \binom{[k]}{d}} \left| \mathbb{E}_{\mathbf{x} \in V^{[k]}} \left[f(\mathbf{x}) \prod_{B \in \binom{[k]}{d}} u_B(\mathbf{x}_B) \right] \right|$$

We say that a k-graph H of edge density p is ϵ-quasirandom of order $d+1$ if $\| H - p \|_{\square^k_d} \leq \epsilon$.

Recap: Octahedral norms

There is another norm which measures (strong) quasirandomness, called the octahedral norm.

Definition: Given $f : V^k \to \mathbb{R}$, we define its *octahedral norm* by

$$\|f\|_{\mathrm{OCT}^k} := \mathbb{E}_{\mathbf{x}^{(0)},\mathbf{x}^{(1)}\in V^k}\left[\prod_{\omega\in\{0,1\}^k} f(\mathbf{x}^{(\omega)})\right]^{1/2^k},$$

where we write $\mathbf{x}^{(\omega)} := (x_i^{(\omega_i)})_{i\in[k]}$.

Note that the product is along all edges of the k*-octahedron*, so $\|f\|_{\mathrm{OCT}^k}^{2^k} = t(\mathrm{OCT}^{(k)}, f)$ is the weighted count of k-octahedra.

Example: When $k = 2$ the octahedron is the 4-cycle:

$$V(\mathrm{OCT}^{(2)}) = \{x^{(0)}, x^{(1)}, y^{(0)}, y^{(1)}\}$$
$$E(\mathrm{OCT}^{(2)}) = \{\{x^{(0)}, y^{(0)}\}, \{x^{(0)}, y^{(1)}\}, \{x^{(1)}, y^{(0)}\}, \{x^{(1)}, y^{(1)}\}\}$$

Recap: Octahedral norms

There is another norm which measures (strong) quasirandomness, called the octahedral norm.

Definition: Given $f : V^k \to \mathbb{R}$, we define its *octahedral norm* by

$$\|f\|_{\text{OCT}^k} := \mathbb{E}_{\mathbf{x}^{(0)},\mathbf{x}^{(1)} \in V^k}\left[\prod_{\omega \in \{0,1\}^k} f(\mathbf{x}^{(\omega)})\right]^{1/2^k},$$

where we write $\mathbf{x}^{(\omega)} := (x_i^{(\omega_i)})_{i\in[k]}$.

Note that the product is along all edges of the *k-octahedron*, so $\|f\|_{\text{OCT}^k}^{2^k} = t(\text{OCT}^{(k)}, f)$ is the weighted count of k-octahedra.

Example: When $k = 3$ the octahedron is given by:

$$V(\text{OCT}^{(3)}) = \{x^{(0)}, x^{(1)}, y^{(0)}, y^{(1)}, z^{(0)}, z^{(1)}\}$$
$$E(\text{OCT}^{(3)}) = \{\{x^{(\omega_1)}, y^{(\omega_2)}, z^{(\omega_3)}\} : \omega_1, \omega_2, \omega_3 \in \{0,1\}\}$$

Recap: Octahedral norms

This norm has an associated *inner product of order k*, defined for 2^k functions $f_\omega : V^k \to \mathbb{R}$, $\omega \in \{0,1\}^k$, by

$$\langle (f_\omega)_{\omega \in \{0,1\}^k} \rangle_{\mathrm{OCT}^k} := \mathbb{E}_{\mathsf{x}^{(0)},\mathsf{x}^{(1)} \in V^k} \left[\prod_{\omega \in \{0,1\}^k} f_\omega(\mathsf{x}^{(\omega)}) \right]$$

With this inner product, we have $\| f \|_{\mathrm{OCT}^k}^{2^k} = \langle f, f, \ldots, f \rangle_{\mathrm{OCT}^k}$

Lemma (Gowers-Cauchy-Schwarz inequality):

For any collection of functions $f_\omega : V^k \to \mathbb{R}$, $\omega \in \{0,1\}^k$, we have

$$\langle (f_\omega)_{\omega \in \{0,1\}^k} \rangle_{\mathrm{OCT}^k} \leq \prod_{\omega \in \{0,1\}^k} \| f_\omega \|_{\mathrm{OCT}^k}$$

Corollary: For all $f : V^k \to \mathbb{R}$ we have that $\| f \|_{\square^k_{k-1}} \leq \| f \|_{\mathrm{OCT}^k}$.

Recap: Quasirandomness in additive groups

Definition: Let G be an additive group and $k \geq 2$. Given $f : G \to \mathbb{R}$, we define its *uniformity norm of degree k* by

$$\| f \|_{U^k} := \mathbb{E}_{x,h_1,\ldots,h_k \in G} \left[\prod_{\omega \in \{0,1\}^k} f(x + \omega \cdot \mathrm{h}) \right]^{1/2^k}$$

where $\omega \cdot \mathrm{h} := \omega_1 h_1 + \cdots + \omega_k h_k$.

A set $A \subseteq G$ of density α is ϵ-uniform of degree k if $\| A - \alpha \|_{U^{k+1}} \leq \epsilon$.

Note: This is the weighted count of k-dimensional parallelepipeds.

Example: Linear uniformity ($k = 2$):

$$\| f \|_{U^2}^4 := \mathbb{E}_{x,h_1,h_2 \in G}[f(x)f(x + h_1)f(x + h_2)f(x + h_1 + h_2)]$$

Recap: Quasirandomness in additive groups

Definition: Let G be an additive group and $k \geq 2$. Given $f : G \to \mathbb{R}$, we define its *uniformity norm of degree k* by

$$\| f \|_{U^k} := \mathbb{E}_{x,h_1,\ldots,h_k \in G} \left[\prod_{\omega \in \{0,1\}^k} f(x + \omega \cdot \mathrm{h}) \right]^{1/2^k}$$

where $\omega \cdot \mathrm{h} := \omega_1 h_1 + \cdots + \omega_k h_k$.

A set $A \subseteq G$ of density α is ϵ-uniform of degree k if $\| A - \alpha \|_{U^{k+1}} \leq \epsilon$.

Note: This is the weighted count of k-dimensional parallelepipeds.

Example: Quadratic uniformity ($k = 3$):

$$\begin{aligned} \| f \|_{U^3}^8 := \mathbb{E}_{x,h_1,h_2,h_3 \in G} [& f(x) f(x+h_1) f(x+h_2) f(x+h_1+h_2) \\ & \times f(x+h_3) f(x+h_1+h_3) f(x+h_2+h_3) f(x+h_1+h_2+h_3)] \end{aligned}$$

Connecting the uniformity and octahedral norms

Definition: Given an integer k and an additive group G, we denote by $s : G^k \to G$ its *summing operator*

$$s(x_1, x_2, \ldots, x_k) := x_1 + x_2 + \cdots + x_k$$

Remark: The indicator function of the Cayley hypergraph $H^{(k)}A$ can be written as $A \circ s$ on G^k.

Lemma (Relationship between the U^k and $\textsc{Oct}^k$ norms):

For every real function $f : G \to \mathbb{R}$, we have that $\| f \circ s \|_{\textsc{Oct}^k} = \| f \|_{U^k}$.

From this it easily follows that $H^{(k)}A$ is ϵ-quasirandom of order $k-1$ whenever A is ϵ-uniform of degree $k-1$.

Connecting quasirandomness and uniformity

Theorem:

Let G be a finite additive group and $A \subseteq G$ be a subset.

a) If A is ϵ-uniform of degree d, then for all $k \geq d+1$ the Cayley hypergraph $H^{(k)}A$ is ϵ-quasirandom of order d.
b) Conversely, if $H^{(d+1)}A$ is ϵ-quasirandom of order d, then A is $2\epsilon^{1/2^{d+1}}$-uniform of degree d.

The proof of the theorem relies on the *translation invariance* of the properties considered:

Definition: Given $a \in G$, we define the *translation operator* T^a on $\mathbb{R}^G$ by $T^a f(x) := f(x+a)$.

Claim: $\|H^{(k)}T^a A - \alpha\|_{\square_d^k} = \|H^{(k)}A - \alpha\|_{\square_d^k}$, and $\|T^a f\|_{U^d} = \|f\|_{U^d}$.

Counting subhypergraphs

From this theorem and the counting lemma proven last lecture, we can count all d-linear hypergraphs inside Cayley hypergraphs of sets that are uniform of degree d.

The extra symmetries satisfied by Cayley hypergraphs imply that actually *more* is true.

Definition: Given $1 \leq d < k$, we say that a k-graph F is $(d+1)$-*simple* if the following is true: for every edge $e \in F$, there exists a set of $d+1$ vertices $\{v_1, \dots, v_{d+1}\} \subseteq e$ which is not contained in any other edge of F (i.e. $\{v_1, \dots, v_{d+1}\} \nsubseteq e' \ \ \forall e' \in F \setminus \{e\}$).

We denote the set of all $(d+1)$-simple k-graphs by $\mathcal{S}_{d+1}^{(k)}$.

$(d+1)$-simple hypergraphs

Recall a k-graph F is $(d+1)$-*simple* if: for every edge $e \in F$, there exists a set of $d+1$ vertices not contained in any other edge of F.

All d-linear hypergraphs are $(d+1)$-simple, but the converse is false.

Examples:

- Let F be the connected k-graph on $2k-d$ vertices and two edges (considered last lecture). Then F is only d-linear, but is 1-simple.
- The *squashed octahedron* $\mathrm{OCT}_d^{(k)}$ is defined by:

$$V = \left\{x_1^{(0)}, x_1^{(1)}, \ldots, x_d^{(0)}, x_d^{(1)}, y_1, \ldots, y_{k-d}\right\}$$

$$E = \left\{\{x_1^{(\omega_1)}, \ldots, x_d^{(\omega_d)}, y_1, \ldots, y_{k-d}\} : \omega \in \{0,1\}^d\right\}$$

It is only $(k-1)$-linear, but is d-simple.

Theorem:

Let $A \subseteq G$ be a set of density α in G. Then for every $k \geq d+1$ the following statements are asymptotically equivalent:

(i) A is uniform of degree d: $\quad \|A - \alpha\|_{U^{d+1}} \leq c_1$

(ii) $H^{(k)}A$ correctly counts all $(d+1)$-simple k-graphs:

$$t(F, H^{(k)}A) = \alpha^{|F|} \pm c_2|F| \qquad \forall F \in \mathcal{S}_{d+1}^{(k)}$$

(iii) $H^{(k)}A$ correctly counts $\textsc{Oct}_{d+1}^{(k)}$:

$$t(\textsc{Oct}_{d+1}^{(k)}, H^{(k)}A) = \alpha^{2^{d+1}} \pm c_3$$

Final remarks

The ability to count all hypergraphs in $\mathcal{S}_{d+1}^{(k)}$ is due to some extra symmetries satisfied by Cayley hypergraphs.

It can be shown that these (approximate) symmetries are in fact *necessary and sufficient* for a large hypergraph to have the "correct" count of all $(d+1)$-simple hypergraphs, or even only of $\textsc{Oct}_{d+1}^{(k)}$.

We can similarly study other hypergraphs associated to additive sets by systems of linear equations. Our methods then allow us to count the number of solutions to such systems inside the set considered.

The methods and results showed can be used to analyze arbitrary objects, by the application of *decomposition theorems*.