Quasirandomness notes

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1 Quasirandom graphs

Theorem 1 (Chung, Graham, Wilson)**.** *Let G be a graph with n vertices and edge density p. Then the following statements are asymptotically equivalent:*

(i) G has low discrepancy*:* $\|G - p\|_\square \leq c_1$

(ii) G correctly counts all graphs*:* $t(F, G) = p^{|F|} \pm c_2|F|$ *for all graphs* F

(iii) G has few 4-cycles*:* $t(C_4, G) \leq p^4 + c_3$

(iv) Only the first eigenvalue matters*:* $\lambda_1 = (p \pm c_4)n$*,* $|\lambda_2| \leq c_4 n$

Proof. *(i)* $\Rightarrow$ *(ii)*

Let m be the number of vertices of the graph F, and assume $V(F) = [m]$ and $E(F) = \{e_1, \ldots, e_{|F|}\}$. For $t = 1, \ldots, |F|$, let i_t, j_t be the endpoints of the edge e_t. Then $|t(F, G) - p^{|F|}|$ can be rewritten as

$$\begin{aligned}
&\left| \mathbb{E}_{x_1,\ldots,x_m \in V(G)} \left[\prod_{ij \in E(F)} G(x_i, x_j) - p^{|F|} \right] \right| \\
&\qquad = \left| \mathbb{E}_{x_1,\ldots,x_m \in V(G)} \left[\sum_{t=1}^{|F|} p^{t-1} \left(G(x_{i_t}, x_{j_t}) - p \right) \prod_{s=t+1}^{|F|} G(x_{i_s}, x_{j_s}) \right] \right| \\
&\qquad\quad \leq \sum_{t=1}^{|F|} p^{t-1} \left| \mathbb{E}_{x_1,\ldots,x_m \in V(G)} \left[\left(G(x_{i_t}, x_{j_t}) - p \right) \prod_{s=t+1}^{|F|} G(x_{i_s}, x_{j_s}) \right] \right|
\end{aligned}$$

Take any term in this sum, and for notational convenience assume that $i_t = 1$ and $j_t = 2$. Then for any fixed $x_3, \ldots, x_m \in V(G)$ we have

$$\begin{aligned}
&\left| \mathbb{E}_{x_1,x_2 \in V(G)} \left[\left(G(x_1, x_2) - p \right) \prod_{s=t+1}^{|F|} G(x_{i_s}, x_{j_s}) \right] \right| \\
&\qquad\qquad\qquad = \left| \mathbb{E}_{x_1,x_2 \in V(G)} \left[\left(G(x_1, x_2) - p \right) a_t(x_1) b_t(x_2) \right] \right|,
\end{aligned}$$

where a_t and b_t are the functions given by

$$a_t(x_1) := \prod_{\substack{s>t \\ 1\in e_s}} G(x_{i_s}, x_{j_s}) \quad \text{and} \quad b_t(x_2) := \prod_{\substack{s>t \\ 1\notin e_s}} G(x_{i_s}, x_{j_s})$$

By hypothesis $\|G - p\|_\square \leq c_1$, so the expression on the right is at most c_1 for all fixed $x_3, \cdots, x_m$. Thus

$$\left| \mathbb{E}_{x_1,\ldots,x_m \in V(G)} \left[(G(x_{i_t}, x_{j_t}) - p) \prod_{s=t+1}^{|F|} G(x_{i_s}, x_{j_s}) \right] \right| \leq c_1,$$

implying $|t(F,G) - p^{|F|}| \leq c_1|F|$, and so we may take $c_2 = c_1$.

(ii) $\Rightarrow$ *(iii)*
This is just a special case, and we can take $c_3 = 4c_2$.

(iii) $\Rightarrow$ *(iv)*
Suppose the vertices of G are labelled by $\{1, 2, \ldots, n\}$, and denote the adjacency matrix of G by A. First note that $\lambda_1 \geq pn$, since

$$|\lambda_1| = \max_{v\neq 0} \frac{\|Av\|_2}{\|v\|_2} \geq \frac{\|Ae\|_2}{\|e\|_2} \geq \frac{e^t Ae}{\|e\|_2^2} = \frac{pn^2}{n} = pn,$$

where $e = (1, 1, \ldots, 1)^t$.

Now we note that, for any $k \in \mathbb{N}$, the entry (i,j) on the matrix A^k counts the number of (directed) paths of length k on G beginning at vertex i and ending at vertex j. For instance,

$$(A^3)_{ij} = \sum_{k,\ell=1}^{n} A_{ik}A_{k\ell}A_{\ell j} = |\{(k,\ell) \in [n]^2 : ik, k\ell, \ell j \in E(G)\}|$$

is the number of directed paths of length 3 starting at i and ending at j.

Thus $(A^4)_{ii}$ counts the number of labelled 4-cycles starting (and ending) at vertex i. This implies that

$$t(C_4, G) = \frac{1}{n^4}\sum_{i=1}^{n}(A^4)_{ii} = \frac{1}{n^4}tr(A^4) = \frac{1}{n^4}\sum_{i=1}^{n}\lambda_i^4 \geq \frac{\lambda_1^4 + \lambda_2^4}{n^4}$$

By assumption $t(C_4, G) \leq p^4 + c_3$, which together with $\lambda_1 \geq pn$ implies that $\lambda_1 \leq (p + c_3^{1/4})n$ and $|\lambda_2| \leq c_3^{1/4}n$. We may then take $c_4 = c_3^{1/4}$.

(iv) $\Rightarrow$ *(i)*
We will first show that $\|A - pJ\|_{sp} \leq 6c_4^{1/2}n$, where A is the adjacency matrix of G, $J = ee^t$ is the $n \times n$ all-ones matrix and $\|\cdot\|_{sp}$ is the spectral norm.

Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of eigenvectors of A, where v_i is an eigenvector associated to the eigenvalue λ_i for all $1 \leq i \leq n$.

If we suppose the graph G is regular of degree pn, then the result we want to prove is simple: in this case $e_1 := e/\sqrt{n}$ is a unitary eigenvector of A with eigenvalue $\lambda_1 = pn$, and so

$$A - pJ = A - pne_1e_1^t = \sum_{j=2}^{n} \lambda_j v_j v_j^t$$

has spectral norm equal to $|\lambda_2| \leq c_4 n$.

If we do not suppose G is regular, then we can decompose

$$\begin{aligned} A - pJ &= \sum_{i=1}^{n} \lambda_i v_i v_i^t - pne_1e_1^t \\ &= \lambda_1 v_1 v_1^t - pnv_1v_1^t + \sum_{j=2}^{n} \lambda_j v_j v_j^t + pnv_1v_1^t - pne_1e_1^t \\ &= M_1 + M_2 + M_3, \end{aligned}$$

where

$$M_1 = (\lambda_1 - pn)v_1v_1^t, \quad M_2 = \sum_{j=2}^{n} \lambda_j v_j v_j^t, \quad M_3 = pn(v_1v_1^t - e_1e_1^t).$$

Clearly $\|M_1\|_{sp} = |\lambda_1 - pn| \leq c_4 n$ and $\|M_2\|_{sp} = |\lambda_2| \leq c_4 n$.

Let us now bound $\|M_3\|_{sp}$. Since M_3 is symmetric real, we know that

$$\|M_3\|_{sp} = \max_{\|u\|_2 = 1} |u^t M_3 u|.$$

Moreover, for any fixed $u \in \mathbb{R}^n$ with $\|u\|_2 = 1$ we have that

$$\begin{aligned} |u^t M_3 u| &= pn \cdot |(u^t v_1)^2 - (u^t e_1)^2| \\ &= pn \cdot |u^t(v_1 + e_1)| \cdot |u^t(v_1 - e_1)| \\ &\leq 2pn \cdot \|v_1 - e_1\|_2, \end{aligned}$$

where the last inequality follows from Cauchy-Schwarz. It thus suffices to bound $\|v_1 - e_1\|_2$.

Decompose $e_1 = \mu v_1 + w$, where $\mu = e_1^t v_1$ and w is orthogonal to v_1. Note that $\|w\|_2 \leq 1$ (by Pythagoras' theorem) and that, up to changing v_1 by $-v_1$, we can assume $\mu \geq 0$. Then

$$\begin{aligned} pn = e_1^t A e_1 \leq \|Ae_1\|_2 &= \|A(\mu v_1 + w)\|_2 \\ &\leq \mu\|Av_1\|_2 + \|Aw\|_2 \\ &\leq \mu\lambda_1 + |\lambda_2| \cdot \|w\|_2 \\ &\leq \mu(p + c_4)n + c_4 n \end{aligned}$$

$$\Rightarrow \quad e_1^t v_1 = \mu \geq \frac{p - c_4}{p + c_4} \geq 1 - \frac{2c_4}{p}$$

From this we conclude that

$$\|v_1 - e_1\|_2^2 = v_1^t v_1 - 2v_1^t e_1 + e_1^t e_1 = 2(1 - v_1^t e_1) \leq \frac{4c_4}{p}$$

We thus conclude that $\|M_3\|_{sp} \leq 2pn \cdot \|v_1 - e_1\|_2 \leq 4c_4^{1/2} n$ and

$$\|A - pJ\|_{sp} \leq \|M_1\|_{sp} + \|M_2\|_{sp} + \|M_3\|_{sp} \leq 6c_4^{1/2} n,$$

as wished.

The rest follows immediately from Cauchy-Schwarz. Indeed, for any subsets $X, Y \subseteq V(G)$ we have

$$\begin{aligned} \frac{1}{n^2}\left|\sum_{x\in X}\sum_{y\in Y}(G(x,y)-p)\right| &= \frac{1}{n^2}\left|\sum_{i=1}^{n}\sum_{j=1}^{n}(A_{ij}-p)\mathbf{1}_X(i)\mathbf{1}_Y(j)\right| \\ &\leq \frac{1}{n^2}\|A-pJ\|_{sp}\|\mathbf{1}_X\|_2\|\mathbf{1}_Y\|_2 \\ &\leq 6c_4^{1/2} \end{aligned}$$

We thus obtain property (i) with $c_1 = 6c_4^{1/2}$. $\square$

2 Uniformity and quasirandomness in additive groups

Recall that we defined the U^2 norm of a function $f : G \to \mathbb{R}$ by

$$\|f\|_{U^2(G)} := \mathbb{E}_{x,h_1,h_2\in G}\,[\,f(x)f(x+h_1)f(x+h_2)f(x+h_1+h_2)\,]^{1/4}$$

Lemma 1. *For all real functions $f : G \to \mathbb{R}$, we have $\|f\|_{U^2(G)} = \|\hat{f}\|_{\ell^4(\hat{G})}$.*

Proof. Since f is real-valued, for all $\gamma \in \hat{G}$ we have that

$$\begin{aligned}|\hat{f}(\gamma)|^4 &= \mathbb{E}_{x\in G}[f(x)\overline{\gamma(x)}]\,\mathbb{E}_{y\in G}[f(y)\overline{\gamma(y)}]\,\mathbb{E}_{z\in G}[f(z)\gamma(z)]\,\mathbb{E}_{w\in G}[f(w)\gamma(w)]\\ &= \mathbb{E}_{x,y,z,w\in G}\,[f(x)f(y)f(z)f(w)\gamma(-x-y+z+w)]\end{aligned}$$

Using the *orthogonality relations of characters* we then obtain

$$\begin{aligned}\|\hat{f}\|^4_{\ell^4(\hat{G})} &= \mathbb{E}_{x,y,z,w\in G}\left[f(x)f(y)f(z)f(w)\sum_{\gamma\in\hat{G}}\gamma(-x-y+z+w)\right]\\ &= \mathbb{E}_{x,y,z,w\in G}\left[f(x)f(y)f(z)f(w)\cdot|G|\mathbf{1}_{\{x+y=z+w\}}\right]\end{aligned}$$

Now we note that, when x, h_1, h_2 are uniformly distributed over G, the quadruple $(x, x+h_1+h_2, x+h_1, x+h_2)$ is uniformly distributed over all solutions in G to $x+y=z+w$. This implies that the last expression is equal to

$$\mathbb{E}_{x,h_1,h_2\in G}\,[\,f(x)f(x+h_1+h_2)f(x+h_1)f(x+h_2)\,] \;=\; \|f\|^4_{U^2(G)},$$

finishing the proof. □

We defined in the lecture the following *inner product of second order*:

$$\langle f_1,f_2,f_3,f_4\rangle_{U^2(G)} = \mathbb{E}_{x,h_1,h_2\in G}\,[\,f_1(x)f_2(x+h_1)f_3(x+h_2)f_4(x+h_1+h_2)\,]$$

With this inner product we have that $\|f\|_{U^2(G)} = \langle f,f,f,f\rangle^{1/4}_{U^2(G)}$.

Lemma 2 (Gowers-Cauchy-Schwarz inequality)**.** *For any functions $f_1, f_2, f_3, f_4 : G \to \mathbb{R}$ we have*

$$\langle f_1,f_2,f_3,f_4\rangle_{U^2(G)} \le \|f_1\|_{U^2(G)}\|f_2\|_{U^2(G)}\|f_3\|_{U^2(G)}\|f_4\|_{U^2(G)}$$

Proof. By the usual Cauchy-Schwarz inequality applied to the variable h_1, we see that

$$\begin{aligned}\langle f_1,f_2,f_3,f_4\rangle^2_{U^2} &= \mathbb{E}_{x,h_1,h_2\in G}\,[\,f_1(x)f_2(x+h_1)f_3(x+h_2)f_4(x+h_1+h_2)\,]^2\\ &= \mathbb{E}_{h_1}\,[\,\mathbb{E}_x\,[f_1(x)f_2(x+h_1)]\cdot\mathbb{E}_y\,[f_3(y)f_4(y+h_1)]\,]^2\\ &\le \left(\mathbb{E}_{h_1}\left[\mathbb{E}_x\,[f_1(x)f_2(x+h_1)]^2\right]\right)\cdot\left(\mathbb{E}_{h_1}\left[\mathbb{E}_y\,[f_3(y)f_4(y+h_1)]^2\right]\right)\end{aligned}$$

$$= \langle f_1, f_2, f_1, f_2\rangle_{U^2}\langle f_3, f_4, f_3, f_4\rangle_{U^2}$$

Applying the same argument to the variable h_2 instead of h_1, we obtain

$$\langle f_1, f_2, f_3, f_4\rangle_{U^2}^2 \le \langle f_1, f_1, f_3, f_3\rangle_{U^2}\langle f_2, f_2, f_4, f_4\rangle_{U^2}$$

We conclude by using both inequalities one after the other:

$$\begin{aligned}\langle f_1, f_2, f_3, f_4\rangle_{U^2}^4 &\le \langle f_1, f_2, f_1, f_2\rangle_{U^2}^2\langle f_3, f_4, f_3, f_4\rangle_{U^2}^2\\ &\le \langle f_1, f_1, f_1, f_1\rangle_{U^2}\langle f_2, f_2, f_2, f_2\rangle_{U^2}\langle f_3, f_3, f_3, f_3\rangle_{U^2}\langle f_4, f_4, f_4, f_4\rangle_{U^2}\\ &= \|f_1\|_{U^2}^4\|f_2\|_{U^2}^4\|f_3\|_{U^2}^4\|f_4\|_{U^2}^4\end{aligned}$$

□

Theorem 2 (Chung, Graham)**.** *Let G be an additive group of order n and let $A \subseteq G$ be a set of size $|A| = \alpha n$. Then the following are asymptotically equivalent:*

(i) Fourier coefficients: $|\hat{A}(\gamma)| \le c_1$ *for all non-trivial characters* γ

(ii) Additive quadruples: *There are at most $(\alpha^4 + c_2)n^3$ solutions in A of the equation $x + y = z + w$*

(iii) Strong translation: *For all sets $B \subseteq G$, all but at most $c_3 n$ elements $x \in G$ satisfy $|A \cap (B + x)| = \alpha|B| \pm c_3 n$*

(iv) Weak translation: *All but at most $c_4 n$ elements $x \in G$ satisfy $|A \cap (A + x)| = \alpha^2 n \pm c_4 n$*

Proof. First of all, we note that condition *(ii)* is equivalent to saying that $\|A\|_{U^2(G)}^4 \le \alpha^4 + c_2$.

(i) ⇒ *(ii)*
Suppose $|\hat{A}(1)| = \alpha$ and $|\hat{A}(\gamma)| \le c_1$ for all $\gamma \in \hat{G} \setminus \{1\}$. Then

$$\begin{aligned}\|A\|_{U^2(G)}^4 = \|\hat{A}\|_{\ell^4(\hat{G})}^4 &= \alpha^4 + \sum_{\gamma\ne 1}|\hat{A}(\gamma)|^4\\ &\le \alpha^4 + \sum_{\gamma\ne 1} c_1^2|\hat{A}(\gamma)|^2\\ &\le \alpha^4 + c_1^2\|\hat{A}\|_{\ell^2(\hat{G})}^2\\ &= \alpha^4 + c_1^2\|A\|_{L^2(G)}^2 \le \alpha^4 + c_1^2,\end{aligned}$$

so we can take $c_2 = c_1^2$.

(ii) ⇒ *(i)*
If $\|A\|_{U^2(G)}^4 \le \alpha^4 + c_2$, then

$$\alpha^4 + c_2 \ge \|\hat{A}\|_{\ell^4(\hat{G})}^4 = \alpha^4 + \sum_{\gamma\ne 1}|\hat{A}(\gamma)|^4 \ge \alpha^4 + \max_{\gamma\ne 1}|\hat{A}(\gamma)|^4$$

This implies that $\max_{\gamma\neq 1}|\hat{A}(\gamma)| \le c_2^{1/4}$, so we can take $c_1 = c_2^{1/4}$.

(ii) $\Rightarrow$ *(iii)*
We first note that, for any fixed $x \in G$, we have

$$|A \cap (B - x)| - \alpha|B| \;=\; \sum_{y\in G}(A(y) - \alpha)B(x + y)$$

Using this identity and the Gowers-Cauchy-Schwarz inequality we see that

$$\begin{aligned}
\sum_{x\in G}\left(|A \cap (B - x)| - \alpha|B|\right)^2 &= \sum_{x\in G}\left(\sum_{y\in G}(A(y) - \alpha)B(x + y)\right)^2\\
&= \sum_{x,y,z\in G}(A(y) - \alpha)B(x + y)(A(z) - \alpha)B(x + z)\\
&= n^3\langle A - \alpha,\, B,\, A - \alpha,\, B\rangle_{U^2(G)}\\
&\le n^3\|A - \alpha\|^2_{U^2(G)}
\end{aligned}$$

Defining the function $f(x) := A(x) - \alpha$ in G, we easily see that $\hat{f}(1) = 0$ and $\hat{f}(\gamma) = \hat{A}(\gamma)$ for all $\gamma \in \hat{G}\setminus\{1\}$. Supposing $\|A\|^4_{U^2(G)} \le \alpha^4 + c_2$, we then obtain

$$\|A - \alpha\|^4_{U^2(G)} = \|\hat{f}\|^4_{\ell^4(\hat{G})} = \|\hat{A}\|^4_{\ell^4(\hat{G})} - \alpha^4 = \|A\|^4_{U^2(G)} - \alpha^4 \le c_2$$

If less than $(1 - c_3)n$ values $x \in G$ satisfy $|A \cap (B - x)| = \alpha|B| \pm c_3 n$, then

$$\sum_{x\in G}\left(|A \cap (B - x)| - \alpha|B|\right)^2 > c_3 n \cdot (c_3 n)^2 = c_3^3 n^3$$

It thus suffices to take $c_3 = c_2^{1/6}$ for this last inequality to be incompatible with our previous bound.

(iii) $\Rightarrow$ *(iv)*
This is just a special case, and we may take $c_4 = c_3$.

(iv) $\Rightarrow$ *(ii)*
As in the proof that *(ii)* $\Rightarrow$ *(iii)*, we see that

$$\sum_{x\in G}|A \cap (A + x)|^2 = n^3\|A\|^4_{U^2(G)}$$

Assuming $c_4 \le 1$ (as otherwise we may just take $c_2 = 1$), we then have

$$\begin{aligned}
n^3\|A\|^4_{U^2(G)} &\le n \cdot (\alpha^2 + c_4)^2 n^2 + c_4 n \cdot n^2\\
&\le (\alpha^4 + 3c_4)n^3 + c_4 n^3 = (\alpha^4 + 4c_4)n^3
\end{aligned}$$

We may then take $c_2 = 4c_4$. $\square$

Definition. Given a subset $A \subseteq G$, we define its Cayley graph Γ_A by

$$V(\Gamma_A) = G, \quad E(\Gamma_A) = \{xy : x + y \in A\}$$

Lemma 3. *A set $A \subseteq G$ is linear uniform if and only if its Cayley graph Γ_A is quasirandom.*

Proof. We will show that property (iii) of quasirandom graphs applied to Γ_A is the same as property (ii) of quasirandom sets applied to A. Indeed,

$$\begin{aligned} t(C_4, \Gamma_A) &= \mathbb{E}_{a,b,c,d \in G} \left[\Gamma_A(a,b)\Gamma_A(b,c)\Gamma_A(c,d)\Gamma_A(d,a)\right] \\ &= \mathbb{E}_{a,b,c,d \in G} \left[A(a+b)A(b+c)A(c+d)A(d+a)\right] \end{aligned}$$

Let us now make the change of variables $x := a+b$, $h_1 := c-a$, $h_2 := d-b$. It is easy to see that x, h_1, h_2 are uniformly distributed on G, so the last expression is equal to

$$\mathbb{E}_{x,h_1,h_2 \in G} \left[A(x)A(x+h_1)A(x+h_1+h_2)A(x+h_2)\right] = \|A\|_{U^2(G)}^4$$

Thus $t(C_4, \Gamma_A) \leq \alpha^4 + \epsilon$ if and only if $\|A\|_{U^2(G)}^4 \leq \alpha^4 + \epsilon$, as wished. □

Lemma 4. *Let G be an additive group of odd order and suppose $A \subseteq G$ is linear ϵ-uniform. Then there are between $(\alpha^3 - \epsilon)n^2$ and $(\alpha^3 + \epsilon)n^2$ 3-term arithmetic progressions in A.*

Proof. We will use the identity

$$\mathbb{E}_{x,r \in G}[f_1(x)f_2(x+r)f_3(x+2r)] = \sum_{\gamma \in \hat{G}} \widehat{f_1}(\gamma)\widehat{f_2}(\gamma^{-2})\widehat{f_3}(\gamma),$$

where γ^{-2} is the character satisfying $\gamma^{-2}(x) = \gamma(x)^{-2}$ for all $x \in G$. Indeed, the last sum is equal to

$$\begin{aligned} &\sum_{\gamma \in \hat{G}} \mathbb{E}_{x \in G}[f(x)\overline{\gamma(x)}]\, \mathbb{E}_{y \in G}[f(y)\overline{\gamma^{-2}(y)}]\, \mathbb{E}_{z \in G}[f(z)\overline{\gamma(z)}] \\ &\qquad = \sum_{\gamma \in \hat{G}} \mathbb{E}_{x,y,z \in G}[f_1(x)f_2(y)f_3(z)\gamma(-x+2y-z)] \\ &\qquad = \mathbb{E}_{x,y,z \in G}\left[f_1(x)f_2(y)f_3(z) \cdot |G|\mathbf{1}_{\{x+z=2y\}}\right] \\ &\qquad = \mathbb{E}_{x,r \in G}[f_1(x)f_2(x+r)f_3(x+2r)], \end{aligned}$$

where we used the orthogonality relations of characters for the second equality.

As $\hat{A}(1) = \alpha$, we conclude that

$$\mathbb{E}_{x,r \in G}[A(x)A(x+r)A(x+2r)] = \alpha^3 + \sum_{\gamma \in \hat{G} \setminus \{1\}} \hat{A}(\gamma)^2 \hat{A}(\gamma^{-2})$$

We then bound the absolute value of the last sum by

$$\begin{aligned}\left|\sum_{\gamma\in\hat{G}\setminus\{1\}}\hat{A}(\gamma)^2\hat{A}(\gamma^{-2})\right| &\leq \left(\max_{\gamma\in\hat{G}\setminus\{1\}}\hat{A}(\gamma)\right)\sum_{\gamma\in\hat{G}\setminus\{1\}}|\hat{A}(\gamma)|\cdot|\hat{A}(\gamma^{-2})| \\ &\leq \epsilon\left(\sum_{\gamma\in\hat{G}}|\hat{A}(\gamma)|^2\right)^{1/2}\left(\sum_{\gamma\in\hat{G}}|\hat{A}(\gamma^{-2})|^2\right)^{1/2}\end{aligned}$$

where for the last inequality we used Cauchy-Schwarz and the fact that A is linear ϵ-uniform.

We will next show that $\{\gamma^{-2} : \gamma \in \hat{G}\} = \hat{G}$. Note that this would conclude the proof, since it implies that the right hand side of the last inequality is equal to $\epsilon\|\hat{A}\|^2_{\ell^2(\hat{G})} = \epsilon\|A\|^2_{L^2(G)} \leq \epsilon$. Since clearly $\{\gamma^{-2} : \gamma \in \hat{G}\} \subseteq \hat{G}$, it suffices to show that $\gamma^{-2} \neq \eta^{-2}$ whenever γ and η are distinct characters.

But if $\gamma \neq \eta$ and $\gamma^{-2} = \eta^{-2}$, then $\eta\gamma^{-1}$ is a nontrivial character satisfying $(\eta\gamma^{-1})^2 = 1$. This implies that the order of $\eta\gamma^{-1}$ is 2, which is impossible since it must divide $|\hat{G}| = |G|$ which is odd. This contradiction finishes the proof. $\square$

Remark: The assumption that G has odd order cannot be dropped. To see this, consider the group $\mathbb{F}_2^n$ and any set $A \subset \mathbb{F}_2^n$ of density $0 < \alpha < 1$. Then

$$\mathbb{E}_{x,r\in\mathbb{F}_2^n}[A(x)A(x+r)A(x+2r)] = \mathbb{E}_{x,r\in\mathbb{F}_2^n}[A(x)A(x+r)] = \alpha^2$$

is strictly bigger than $\alpha^3 + \epsilon$ for any $\epsilon < (1-\alpha)\alpha^2$.

3 Quasirandomness in hypergraphs

Lemma 5 (Counting lemma for quasirandomness of order d)**.** *For any k-uniform hypergraph H and any $0 \le p \le 1$, we have that*

$$t(F,H) = p^{|F|} \pm |F| \cdot \|H - p\|_{\square_d^k} \qquad \forall F \in \mathcal{L}_d^{(k)}$$

Proof. Let m be the number of vertices of the graph F, and assume $V(F) = [m]$ and $E(F) = \{e_1, \dots, e_{|F|}\}$. Then we can write as a telescoping sum

$$\begin{aligned}
\left|t(F,H) - p^{|F|}\right| &= \left|\mathbb{E}_{\mathbf{x}\in V(H)^{V(F)}}\left[\prod_{e\in E(F)} H(\mathbf{x}_e) - p^{|F|}\right]\right| \\
&= \left|\mathbb{E}_{\mathbf{x}\in V(H)^{V(F)}}\left[\sum_{i=1}^{|F|} p^{i-1}\left(H(\mathbf{x}_{e_i}) - p\right)\prod_{j=i+1}^{|F|} H(\mathbf{x}_{e_j})\right]\right| \\
&\le \sum_{i=1}^{|F|} p^{i-1}\left|\mathbb{E}_{\mathbf{x}\in V(H)^{V(F)}}\left[\left(H(\mathbf{x}_{e_i}) - p\right)\prod_{j=i+1}^{|F|} H(\mathbf{x}_{e_j})\right]\right|
\end{aligned}$$

The i-th term in the final sum is bounded by $\|H - p\|_{\square_d^k}$. Indeed, if we fix all variables other than $\mathbf{x}_{e_i}$, then all the factors except for $(H(\mathbf{x}_{e_i}) - p)$ have the form $u(\mathbf{x}_f)$ for some set f of size at most d, as f is the intersection of e_i with another edge e_j. So the the expectation can be bounded by $\|H-p\|_{\square_d^k}$ for each term, proving the lemma. $\square$

Given functions $f_\omega : V^k \to \mathbb{R}$, $\omega \in \{0,1\}^k$, we define their *inner product of order k* by

$$\langle (f_\omega)_{\omega\in\{0,1\}^k}\rangle_{\mathrm{OCT}^k} := \mathbb{E}_{\mathbf{x}^{(0)},\mathbf{x}^{(1)}\in V^k}\left[\prod_{\omega\in\{0,1\}^k} f_\omega(\mathbf{x}^{(\omega)})\right],$$

where we write $\mathbf{x}^{(\omega)} := (\mathbf{x}_i^{(\omega_i)})_{i\in[k]}$. With this inner product, we have

$$\|f\|_{\mathrm{OCT}^k}^{2^k} = \langle f, f, \dots, f\rangle_{\mathrm{OCT}^k}$$

Lemma 6 (Gowers-Cauchy-Schwarz inequality)**.**

$$\langle (f_\omega)_{\omega\in\{0,1\}^k}\rangle_{\mathrm{OCT}^k} \le \prod_{\omega\in\{0,1\}^k} \|f_\omega\|_{\mathrm{OCT}^k}$$

The proof of this lemma follows by applying Cauchy-Schwarz k times consecutively, and it will be given next class.

Remark: Using the Gowers-Cauchy-Schwarz inequality it is possible to prove that the octahedral norm satisfies the triangle inequality, and is thus indeed a norm.

Lemma 7. *For every function $f : V^k \to \mathbb{R}$, we have that $\|f\|_{\square^k_{k-1}} \le \|f\|_{\mathrm{OCT}^k}$.*

Proof. For any function $u_B : V^B \to [0,1]$, $B \in \binom{[k]}{k-1}$, we let $f_{\omega_B} : V^{[k]} \to \mathbb{R}$ be the function $f_{\omega_B}(\mathbf{x}_{[k]}) := u_B(\mathbf{x}_B)$, where $\omega_B \in \{0,1\}^{[k]}$ is the indicator vector of the set B.

By denoting $f_{\mathbf{1}} := f$ and $f_\omega := 1$ for $\omega \in \{0,1\}^{[k]} \setminus \{\mathbf{1}\}$ not in $\{\omega_B : B \in \binom{[k]}{k-1}\}$, using the Gowers-Cauchy-Schwarz inequality we conclude that

$$\left|\mathbb{E}_{\mathbf{x}\in V^{[k]}}\left[f(\mathbf{x})\prod_{B\in\binom{[k]}{k-1}} u_B(\mathbf{x}_B)\right]\right| = \left|\mathbb{E}_{\mathbf{x}^{(0)},\mathbf{x}^{(1)}\in V^k}\left[\prod_{\omega\in\{0,1\}^k} f_\omega(\mathbf{x}^{(\omega)})\right]\right| \le \prod_{\omega\in\{0,1\}^k} \|f_\omega\|_{\mathrm{OCT}^k}$$

Since clearly $\|f_\omega\|_{\mathrm{OCT}^k} \le \|f_\omega\|_{\ell^\infty} \le 1$ for all $\omega \in \{0,1\}^{[k]} \setminus \{\mathbf{1}\}$, the last product is at most $\|f\|_{\mathrm{OCT}^k}$. As this inequality is valid for all functions $u_B : V^B \to [0,1]$, $B \in \binom{[k]}{k-1}$, we obtain the claim. ☐

Theorem 3 (Kohayakawa, Rödl, Skokan)**.** *Let H be a k-uniform hypergraph with edge density p. Then the following statements are asymptotically equivalent:*

(i) H is strongly quasirandom: $\|H - p\|_{\square^k_{k-1}} \le c_1$

(ii) H correctly counts all *hypergraphs:*

$$t(F,H) = \alpha^{|F|} \pm c_2|F| \quad \text{for all } k\text{-hypergraphs } F$$

(iii) H correctly counts $\mathrm{OCT}^{(k)}$ *and all its subhypergraphs:*

$$t(F,H) = \alpha^{|F|} \pm c_3 \quad \text{for all } F \subseteq \mathrm{OCT}^{(k)}$$

Proof. *(i)* $\Rightarrow$ *(ii)*

This follows immediately from the counting lemma, and we may take $c_2 = c_1$.

(ii) $\Rightarrow$ *(iii)*

This is a special case, and we may take $c_3 = 2^k c_2$.

(iii) $\Rightarrow$ *(i)*

Suppose $t(F,H) = p^{|F|} \pm c_3$ for all subhypergraphs $F \subseteq \mathrm{OCT}^{(k)}$. Then

$$\begin{aligned} t(\mathrm{OCT}^{(k)}, H-p) &= \mathbb{E}_{\mathbf{x}\in V(H)^{V(\mathrm{OCT}^{(k)})}}\left[\prod_{e\in\mathrm{OCT}^{(k)}} (H(\mathbf{x}_e) - p)\right] \\ &= \mathbb{E}_{\mathbf{x}\in V(H)^{V(\mathrm{OCT}^{(k)})}}\left[\sum_{F\subseteq\mathrm{OCT}^{(k)}} \prod_{e\in F} H(\mathbf{x}_e) \prod_{e\in\mathrm{OCT}^{(k)}\setminus F} (-p)\right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{F \subseteq \textsc{Oct}^{(k)}} \mathbb{E}_{\mathbf{x} \in V(H)^{V(\textsc{Oct}^{(k)})}} \left[\prod_{e \in F} H(\mathbf{x}_e) \right] \cdot (-p)^{2^k - |F|} \\
&= \sum_{F \subseteq \textsc{Oct}^{(k)}} t(F, H) \cdot (-p)^{2^k - |F|} \\
&= \sum_{F \subseteq \textsc{Oct}^{(k)}} (p^{|F|} \pm c_3)(-p)^{2^k - |F|} \\
&= \sum_{F \subseteq \textsc{Oct}^{(k)}} p^{|F|}(-p)^{2^k - |F|} \pm 2^{2^k} c_3 \\
&= \pm 2^{2^k} c_3
\end{aligned}
$$

Thus $\|H - p\|_{\textsc{Oct}^k} = t(\textsc{Oct}^{(k)}, H - p)^{1/2^k} \leq 2c_3^{1/2^k}$, and the claim follows from the inequality $\|H - p\|_{\square^k_{k-1}} \leq \|H - p\|_{\textsc{Oct}^k}$ (with $c_1 = 2c_3^{1/2^k}$). $\square$

4 Comparing quasirandomness in additive groups and in hypergraphs

Lemma 8. *For every function $f : V^k \to \mathbb{R}$, we have that $\|f\|_{\square^k_{k-1}} \leq \|f\|_{\mathrm{OCT}^k}$.*

Proof. For any function $u_B : V^B \to [0,1]$, $B \in \binom{[k]}{k-1}$, we let $f_{\omega_B} : V^{[k]} \to \mathbb{R}$ be the function $f_{\omega_B}(\mathbf{x}_{[k]}) := u_B(\mathbf{x}_B)$, where $\omega_B \in \{0,1\}^{[k]}$ is the indicator vector of the set B.

By denoting $f_{\mathbf{1}} := f$ and $f_\omega := 1$ for $\omega \in \{0,1\}^{[k]} \setminus \{\mathbf{1}\}$ not in $\{\omega_B : B \in \binom{[k]}{k-1}\}$, using the Gowers-Cauchy-Schwarz inequality we conclude that

$$\left|\mathbb{E}_{\mathbf{x}\in V^{[k]}}\left[f(\mathbf{x})\prod_{B\in\binom{[k]}{k-1}} u_B(\mathbf{x}_B)\right]\right| = \left|\mathbb{E}_{\mathbf{x}^{(0)},\mathbf{x}^{(1)}\in V^k}\left[\prod_{\omega\in\{0,1\}^k} f_\omega(\mathbf{x}^{(\omega)})\right]\right|$$
$$\leq \prod_{\omega\in\{0,1\}^k} \|f_\omega\|_{\mathrm{OCT}^k}$$

Since clearly $\|f_\omega\|_{\mathrm{OCT}^k} \leq \|f_\omega\|_{\ell^\infty} \leq 1$ for all $\omega \in \{0,1\}^{[k]} \setminus \{\mathbf{1}\}$, the last product is at most $\|f\|_{\mathrm{OCT}^k}$. As this inequality is valid for all functions $u_B : V^B \to [0,1]$, $B \in \binom{[k]}{k-1}$, we obtain the claim. $\square$

Remark: Denoting by $s_k : G^k \to G$ the *sum operator* $(x_1, \dots, x_k) \mapsto x_1 + \dots + x_k$, one can easily prove the identity $\|f\|_{U^k(G)} = \|f \circ s_k\|_{\mathrm{OCT}^k}$, valid for all functions $f : G \to \mathbb{R}$.

Theorem 4. *Let G be a finite additive group and $A \subseteq G$ be a subset.*

a) If A is ϵ-uniform of degree d, then for all $k \geq d+1$ the Cayley hypergraph $H^{(k)}A$ is ϵ-quasirandom of order d.

b) Conversely, if $H^{(d+1)}A$ is ϵ-quasirandom of order d, then A is $2\epsilon^{1/2^{d+1}}$-uniform of degree d.

Proof. We will prove the theorem more generally for functions $f : G \to [0,1]$ instead of only sets $A \subseteq G$. The statement then follows by taking f to be the indicator function of the set A.

a) Suppose $f : G \to \mathbb{R}$ satisfies $\|f - \alpha\|_{U^{d+1}} \leq \epsilon$, and choose optimal functions $u_B : G^B \to [0,1]$, $B \in \binom{[k]}{d}$, so that

$$\|H^{(k)}f - \alpha\|_{\square^k_d} := \|f \circ s - \alpha\|_{\square^k_d} = \left|\mathbb{E}_{\mathbf{x}\in G^{[k]}}\left[(f(s(\mathbf{x})) - \alpha)\prod_{B\in\binom{[k]}{d}} u_B(\mathbf{x}_B)\right]\right|$$

We now separate the first $d+1$ variables $\mathbf{x}_{[d+1]}$ from the rest, thus writing $\|H^{(k)}f-\alpha\|_{\square^k_d}$ as

$$\left|\mathbb{E}_{\mathbf{x}_{[k]\setminus[d+1]}}\mathbb{E}_{\mathbf{x}_{[d+1]}}\left[\left(f\left(s(\mathbf{x}_{[d+1]})+s(\mathbf{x}_{[k]\setminus[d+1]})\right)-\alpha\right)\prod_{B\in\binom{[k]}{d}}u_B(\mathbf{x}_B)\right]\right|,$$

where the first expectation is over $G^{[k]\setminus[d+1]}$ and the second is over $G^{[d+1]}$.

By using the triangle inequality and choosing suitable functions $v_{D,\mathbf{x}_{[k]\setminus[d+1]}}:G^D\to[0,1]$ for each $\mathbf{x}_{[k]\setminus[d+1]}\in G^{[k]\setminus[d+1]}$ and each set $D\in\binom{[d+1]}{d}$, the last expression is at most

$$\begin{aligned}\mathbb{E}_{\mathbf{x}_{[k]\setminus[d+1]}}&\left|\mathbb{E}_{\mathbf{x}_{[d+1]}}\left[\left(T^{s(\mathbf{x}_{[k]\setminus[d+1]})}f\circ s(\mathbf{x}_{[d+1]})-\alpha\right)\prod_{D\in\binom{[d+1]}{d}}v_{D,\mathbf{x}_{[k]\setminus[d+1]}}(\mathbf{x}_D)\right]\right|\\&\le\mathbb{E}_{\mathbf{x}_{[k]\setminus[d+1]}}\|T^{s(\mathbf{x}_{[k]\setminus[d+1]})}f\circ s-\alpha\|_{\square^{d+1}_d}\\&=\mathbb{E}_{\mathbf{x}_{[k]\setminus[d+1]}}\|f\circ s-\alpha\|_{\square^{d+1}_d}\\&=\|H^{(d+1)}f-\alpha\|_{\square^{d+1}_d}\end{aligned}$$

By the Gowers-Cauchy-Schwarz inequality corollary, this is at most

$$\|H^{(d+1)}f-\alpha\|_{\mathrm{OCT}^{d+1}}=\|H^{(d+1)}(f-\alpha)\|_{\mathrm{OCT}^{d+1}}=\|f-\alpha\|_{U^{d+1}}\le\epsilon,$$

thus showing $\|H^{(k)}f-\alpha\|_{\square^k_d}\le\epsilon$.

b) If $h:V^{d+1}\to[0,1]$, recall that

$$\|h\|^{2^{d+1}}_{\mathrm{OCT}^{d+1}}=\mathbb{E}_{\mathbf{x}^{(0)},\mathbf{x}^{(1)}\in V^{d+1}}\left[\prod_{\omega\in\{0,1\}^k}h(\mathbf{x}^{(\omega)})\right]$$

We can then use the pigeonhole principle to freeze the $\mathbf{x}^{(0)}$ variables and then obtain functions $u_D:V^D\to[0,1]$, $D\in\binom{[d+1]}{d}$, for which

$$\|h\|_{\square^{d+1}_d}\ge\mathbb{E}_{\mathbf{x}\in V^{d+1}}\left[h(\mathbf{x})\prod_{D\in\binom{[d+1]}{d}}u_D(\mathbf{x}_D)\right]\ge\|h\|^{2^{d+1}}_{\mathrm{OCT}^{d+1}}$$

The claim easily follows from the identity $\|f\|_{U^{d+1}(G)}=\|f\circ s_k\|_{\mathrm{OCT}^{d+1}}$. □

Definition: d-simple k-graphs $\mathcal{S}^{(k)}_d$, squashed octahedron $\mathrm{OCT}^{(k)}_d$.

Theorem 5. *Let $A\subseteq G$ be a set of density α in G. Then for every $k\ge d+1$ the following statements are asymptotically equivalent:*

(i) A is uniform of degree d: $\quad \|A-\alpha\|_{U^{d+1}} \le c_1$

(ii) $H^{(k)}A$ correctly counts all hypergraphs in $\mathcal{S}^{(k)}_{d+1}$:

$$t(F, H^{(k)}A) = \alpha^{|F|} \pm c_2|F| \qquad \forall F \in \mathcal{S}^{(k)}_{d+1}$$

(iii) $H^{(k)}A$ correctly counts $\mathrm{OCT}^{(k)}_{d+1}$:

$$t(\mathrm{OCT}^{(k)}_{d+1}, H^{(k)}A) = \alpha^{2^{d+1}} \pm c_3$$

Proof. $(i) \Rightarrow (ii)$:

$$\begin{aligned}
\left|t(F,H^{(k)}f) - \alpha^{|F|}\right| &= \left|\mathbb{E}_{\mathbf{x}_V \in G^V}\left[\sum_{i=1}^{|F|} \alpha^{i-1}\left(f\circ s(\mathbf{x}_{e_i}) - \alpha\right)\prod_{j=i+1}^{|F|} f\circ s(\mathbf{x}_{e_j})\right]\right| \\
&\le \sum_{i=1}^{|F|}\left|\mathbb{E}_{\mathbf{x}_V \in G^V}\left[\left(f\circ s(\mathbf{x}_{e_i}) - \alpha\right)\prod_{j=i+1}^{|F|} f\circ s(\mathbf{x}_{e_j})\right]\right| \\
&\le \sum_{i=1}^{|F|}\mathbb{E}_{\mathbf{x}_{V\setminus f_i}}\left|\mathbb{E}_{\mathbf{x}_{f_i}}\left[\left(f\left(s(\mathbf{x}_{f_i}) + s(\mathbf{x}_{e_i\setminus f_i})\right) - \alpha\right)\right.\right. \\
&\qquad\qquad \left.\left.\times \prod_{j=i+1}^{|F|} f\left(s(\mathbf{x}_{e_j\cap f_i}) + s(\mathbf{x}_{e_j\setminus f_i})\right)\right]\right|
\end{aligned}$$

Let us now consider the i-th term in the last sum. For a fixed $\mathbf{x}_{V\setminus f_i} \in G^{V\setminus f_i}$ and each $i+1 \le j \le |F|$, define the function $u_{j,\mathbf{x}_{V\setminus f_i}} := T^{s(\mathbf{x}_{e_j\setminus f_i})} f\circ s$ on $G^{e_j\cap f_i}$. With this notation, the last sum becomes

$$\begin{aligned}
&\sum_{i=1}^{|F|}\mathbb{E}_{\mathbf{x}_{V\setminus f_i}}\left|\mathbb{E}_{\mathbf{x}_{f_i}}\left[\left(T^{s(\mathbf{x}_{e_i\setminus f_i})} f\circ s(\mathbf{x}_{f_i}) - \alpha\right)\prod_{j=i+1}^{|F|} u_{j,\mathbf{x}_{V\setminus f_i}}(\mathbf{x}_{e_j\cap f_i})\right]\right| \\
&\qquad\qquad \le \sum_{i=1}^{|F|}\mathbb{E}_{\mathbf{x}_{V\setminus f_i}}\left\|T^{s(\mathbf{x}_{e_i\setminus f_i})} f\circ s - \alpha\right\|_{\square^{d+1}_d} \\
&\qquad\qquad = |F|\cdot\|f\circ s - \alpha\|_{\square^{d+1}_d}
\end{aligned}$$

Then item (ii) follows from the corollary of the Gowers-Cauchy-Schwarz inequality, since

$$\|f\circ s-\alpha\|_{\square^{d+1}_d} = \|H^{(k)}f - \alpha\|_{\square^{d+1}_d} \le \|H^{(k)}f-\alpha\|_{\mathrm{OCT}^{d+1}} = \|f-\alpha\|_{U^{d+1}} \le c_1,$$

and we may take $c_2 = c_1$.

$(ii) \Rightarrow (iii)$:
This is a special case, and we may take $c_3 = 2^{d+1}c_2$.

$(iii) \Rightarrow (i)$:
First we note that $t(\textsc{Oct}^{(k)}_{d+1}, H^{(k)}f) = t(\textsc{Oct}^{(d+1)}, H^{(d+1)}f)$ for any function $f : G \to \mathbb{R}$. Indeed, we have that

$$\begin{aligned}
t(\textsc{Oct}^{(k)}_{d+1}, H^{(k)}f) &= \mathbb{E}_{\mathbf{x}\in G^V}\left[\prod_{e\in \textsc{Oct}^{(k)}_{d+1}} f \circ s(\mathbf{x}_e)\right] \\
&= \mathbb{E}_{\mathbf{y}\in G^{[k-d]}}\mathbb{E}_{\mathbf{x}^{(0)},\mathbf{x}^{(1)}\in G^{[d+1]}}\left[\prod_{\omega\in\{0,1\}^{d+1}} f\left(s(\mathbf{y}) + s(\mathbf{x}^{(\omega)})\right)\right] \\
&= \mathbb{E}_{\mathbf{y}\in G^{[k-d]}}\left[t(\textsc{Oct}^{(d+1)}, H^{(d+1)}T^{s(\mathbf{y})}f)\right] \\
&= t(\textsc{Oct}^{(d+1)}, H^{(d+1)}f)
\end{aligned}$$

Item (i) then follows from the hypergraph quasirandomness theorem given last lecture and the item $b)$ of the last theorem. $\square$