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Convex Optimization

Winter Term 2020/21

— Exercise Sheet 3 (November 17, 2020) —

Exercise 3.1. Show: int $\mathcal{S}^n_+ = \mathcal{S}^n_{++}$.

Exercise 3.2. Recall that a complex square matrix $A \in \mathbb{C}^{n\times n}$ is *Hermitian* (or self-adjoint) if $A = A^*$, i.e., $A_{ij} = \overline{A}_{ji}$ for all entries of A. The Hermitian matrices form a real vector space (of dimension n^2), with the Frobenius inner product

$$\langle A, B\rangle = \sum_{ij} \overline{A}_{ij} B_{ij} = \mathrm{Tr}(A^* B).$$

A Hermitian matrix $M \in \mathbb{C}^{n\times n}$ is *positive semidefinite* if $z^* M z \geq 0$ for all $z \in \mathbb{C}^n$, or equivalently if all eigenvalues of M are non-negative.
Consider the set $\mathcal{H}^n_+$ of positive semidefinite complex $n \times n$-matrices as a subset of the Hermitian matrices. Show:

(a) $\mathcal{H}^n_+$ is a self-dual proper convex cone for any $n \geq 1$.

(b) $\mathcal{H}^2_+$ is isometric to $\mathcal{L}^{3+1}$.

Exercise 3.3. Given $x_1, \dots, x_n \in \mathbb{R}$, consider the following matrix

$$X = \begin{pmatrix} 1 & x_1 & \dots & x_n \\ x_1 & x_1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ x_n & 0 & 0 & x_n \end{pmatrix}.$$

That is, $X \in \mathcal{S}^{n+1}$ is the matrix indexed by $\{0, 1, \dots, n\}$, with entries $X_{00} = 1$, $X_{0i} = X_{i0} = X_{ii} = x_i$ for $i \in [n]$, and all other entries are equal to 0.
Use the Schur complement to show:

$$X \succeq 0 \Longleftrightarrow x_i \geq 0 \text{ for all } i \in [n] \text{ and } \sum_{i=1}^n x_i \leq 1.$$

Exercise 3.4. A matrix $X \in \mathcal{S}^n$ is said to be *diagonally dominant* if

$$X_{ii} \geq \sum_{j\in[n]: j\neq i} |X_{ij}| \quad \text{for all } i \in [n].$$

A matrix X is called *scaled diagonally dominant* if there is positive definite diagonal matrix D so that DXD is diagonally dominant.

(a) Show: If X is diagonally dominant, then $X \succeq 0$.

(b) Show: If X is scaled diagonally dominant, then $X \succeq 0$.