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Convex Optimization

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— Exercise Sheet 5 (December 8, 2020) —

Exercise 5.1. Determine the Löwner-John ellipsoid $\mathcal{E}_{\text{in}}(C_n)$ of the regular n-gon C_n in the plane

$$C_n = \text{conv}\{(\cos(2\pi k/n), \sin(2\pi k/n)) \in \mathbb{R}^2 : k = 0, 1, \dots, n-1\}.$$

Exercise 5.2.

(a) Let $P \subseteq \mathbb{R}^n$ be an n-dimensional polytope and let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Show:

$$\mathcal{E}_{out}(AP) = A\mathcal{E}_{out}(P).$$

(b) Let $T \subseteq \mathbb{R}^3$ be a regular tetrahedron with inradius 1. Show that $\mathcal{E}_{in}(T) = B_3$ and that

$$B_3 \subseteq T \subseteq 3B_3.$$

holds.

Exercise 5.3. Let $C \in \mathcal{S}^n$ be a symmetric matrix and let $G = (V, E)$ be a graph with vertex set $V = [n]$. The solution of the following MAXDET problem

$$\begin{aligned} \max \quad & \det\left(C + \sum_{\{i,j\}\in E} x_{ij}E_{ij}\right)^{1/n} \\ & C + \sum_{\{i,j\}\in E} x_{ij}E_{ij} \in \mathcal{S}^n_{\succeq 0}, \end{aligned}$$

is said to be a *G-modification of C with maximal entropy*. Show: If a G-modification of C with maximal entropy $A^* = C + \sum_{\{i,j\}\in E} x^*_{ij}E_{ij}$ exists, then

$$\forall \{i,j\} \in E \ : \ ((A^*)^{-1})_{ij} = 0.$$

Exercise 5.4. Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric polytope ($P = -P$). Find a conic program (with possibly infinitely many constraints) which determines the minimal value $\rho \in \mathbb{R}$ such that there exists an ellipsoid E for which $E \subseteq P \subseteq \rho E$ holds.