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Convex Optimization

Winter Term 2020/21

— Exercise Sheet 7 (January 12, 2021) —

Exercise 7.1. Let $X \in \{\pm 1\}^{n \times n}$ be a symmetric matrix whose entries are 1 or -1. Show that X is positive semidefinite if and only if $X = xx^{\mathsf{T}}$ for some $x \in \{\pm 1\}^n$.

Exercise 7.2. Let $G = (V, E)$ be a graph and $w \in \mathbb{R}^E_+$ a nonnegative weight function on the edges of G.

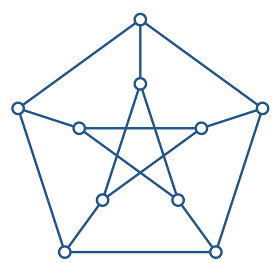
(a) Show that $\mathrm{mc}(G, w) = \mathrm{sdp}(G, w)$ holds when G is a bipartite graph.

(b) Determine the values of $\mathrm{mc}(\mathsf{K}_n, \mathbf{e})$ and $\mathrm{sdp}(\mathsf{K}_n, \mathbf{e})$ for the complete graph K_n on n vertices and the all-ones weight function $\mathbf{e} = (1, \ldots, 1)^{\mathsf{T}}$

Exercise 7.3. Let $A \in \mathcal{S}^n_+$ be a positive semidefinite matrix. Show the following identity:

$$\max\{\langle A, xx^{\mathsf{T}}\rangle : x \in \{-1, +1\}^n\} = \max\left\{\frac{2}{\pi}\langle A, \arcsin X\rangle : X \in \mathcal{S}^n_+,\ X_{ii} = 1, i = 1, \ldots, n\right\}.$$

Exercise 7.4.



(a) Use a computer to solve $\mathrm{sdp}(P, \mathbf{e})$ for the Petersen graph P (see figure), then use your result to determine $\mathrm{mc}(P, \mathbf{e})$.

(b) Let $C_n = (V_n, E_n)$ be the cycle graph with n vertices, defined by

$$V_n = \{1, \ldots, n\} \quad \text{and} \quad E_n = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}.$$

Compute $\mathrm{sdp}(C_n, \mathbf{e})$ for $n = 2k + 1$ and $k \in \{1, 2, 3, 4\}$.