Universität zu Köln
Mathematisches Institut
Prof. Dr. F. Vallentin
G. Fischer
Dr. M.C. Zimmermann

Convex Optimization

Winter Term 2020/21

— Exercise Sheet 8 (January 19, 2021) —

Exercise 8.1. Let $A \in \mathbb{R}^{m\times n}$ be a matrix. Write the optimization problem $\mathrm{sdp}_{\infty\to 1}(A)$ as semidefinite program in primal standard form.

Exercise 8.2. Let $A \in \mathbb{R}^{m\times n}$ be a rectangular matrix. Show:

$$\|A\|_{\infty\to 1} = \max\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}x_iy_j : x_i, y_j \in [-1,1],\ i \in [m], j \in [n]\right\}.$$

Exercise 8.3. Show: If $A \in \mathcal{S}^n_+$, then

$$\|A\|_{\infty\to 1} = \max\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}x_ix_j : x_i \in \{-1,1\},\ i \in [n]\right\}.$$

Exercise 8.4. Show that in the setting of the Grothendieck problem with rank r constraint we have the following inequality

$$\left|\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}\sum_{k=1}^{\infty} f_{2k+1}(u_i\cdot v_j)^{2k+1}\right| \le (1-f_1)\mathrm{SDP}_{m+n}(A),$$

for every matrix $A = (A_{ij}) \in \mathbb{R}^{m\times n}$, where

$$\mathbb{E}\left[\frac{Zu}{\|Zu\|}\cdot\frac{Zv}{\|Zv\|}\right] = \sum_{k=0}^{\infty} f_{2k+1}(u\cdot v)^{2k+1},$$

where u, v are unit vectors, and where $Z \in \mathbb{R}^{r\times(m+n)}$ is a random matrix whose entries are distributed independently according to the standard normal distribution with mean 0 and variance 1.

Hint. Consider the $m+n$ vectors $w_i = u_i$, $w_{m+i} = v_j$. Then the $(m+n)\times(m+n)$-matrix with entries

$$\sum_{k=1}^{\infty} f_{2k+1}(w_i\cdot w_j)^{2k+1}$$

is positive semidefinite. Determine the diagonal entries of this matrix.