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Convex Optimization

Winter Term 2020/21

— Exercise Sheet 9 (January 26, 2021) —

Exercise 9.1. Let $G = (V, E)$ be a graph. Determine the dual of the following semidefinite program and prove that strong duality holds

$$\begin{aligned} \vartheta^+(G) = \max \quad & \langle J, X\rangle \\ & X \succeq 0 \\ & \mathrm{Tr}(X) = 1 \\ & X_{i,j} \leq 0 \text{ for } \{i,j\} \in E. \end{aligned}$$

Exercise 9.2. Let $G = (V, E)$ be a graph. Show:

$$\vartheta(G) = \min\{\lambda_{\max}(Z) \ : \ Z \in \mathcal{S}^V, \ Z = J + T, \ T_{i,j} = 0 \ \text{ if } \{i,j\} \notin E\}.$$

Exercise 9.3. Assume that G is *regular*, i.e. $\mathbf{e} = (1, \ldots, 1)$ is an eigenvector of the adjacency matrix A_G of G. Show:

$$\vartheta(G) \leq |V| \frac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}},$$

where $\lambda_{\min}$ is the smallest and $\lambda_{\max}$ is the largest eigenvalue of A_G.

Exercise 9.4. Determine the Shannon capacity of all graphs having four vertices.