Lemma 1 Falls $(x_k, y_k, s_k) \in \mathcal{F}^\circ$. Dann gilt

a) $(\Delta x_k)^T (\Delta s_k) = 0$

b) $\mu_k = \underbrace{\left(1 - 0.4/\sqrt{n}\right)}_{\sigma} \mu_{k-1}$.

Bew.: a) $\rightarrow$ Aufgabe 13.4

b) $x_{k+1}^T s_{k+1} = (x_k + \Delta x_k)^T (s_k + \Delta s_k)$

$$= x_k s_k + \underbrace{x_k \Delta s_k + \Delta x_k s_k}_{} + \underbrace{\Delta x_k \Delta s_k}_{=0}$$

$$S_k \Delta x_k + X_k \Delta s_k = - X_k S_k e + \sigma \mu_k e$$

Aufsummieren der Einträge $\Longrightarrow$ $s_k^T \Delta x_k + x_k^T \Delta s_k = \underbrace{- x_k^T s_k + \sigma \mu_k n}_{(-1+\sigma) x_k^T s_k}$

$$= \sigma x_k^T s_k. \quad \boxtimes$$

Bsp.: $\mu_0 = 10^6$, $n = 10\,000$, $N = 5\,000$.

Dann $$\mu_N = \left(1 - 0{,}4/\sqrt{10000}\right)^N \mu_0$$

$$= (0{,}996)^N \mu_0 \leq 0{,}002$$

Fehlt nur noch zu zeigen: $(x_k, y_k, s_k) \in \mathcal{F}^0$.

Wir zeigen, dass (x_k, y_k, s_k) sogar in $\mathcal{N}(\theta)$ liegt.

<u>Lemma 2</u> Seien $u, v \in \mathbb{R}^n$, $u^T v \geq 0$. Dann gilt

$$\|UVe\| \leq 2^{-\frac{3}{2}} \|u+v\|^2.$$

<u>Bew.:</u> Es ist

$$0 \leq u^T v = \sum_{u_i v_i > 0} u_i v_i + \sum_{u_i v_i < 0} u_i v_i$$

$$(*) \qquad = \sum_{i \in P} |u_i v_i| - \sum_{i \in M} |u_i v_i| , \quad \text{wobei}$$

$$P = \{ i : u_i v_i > 0 \}, \quad M = \{ i : u_i v_i < 0 \}.$$

Somit $\|UVe\| = \left(\|(u_i v_i)_{i \in P}\|^2 + \|(u_i v_i)_{i \in M}\|^2 \right)^{1/2}$

$$\| \|_2 \leq \| \|_1 \longrightarrow \quad \leq \left(\|(u_i v_i)_{i \in P}\|_1^2 + \|(u_i v_i)_{i \in M}\|_1^2 \right)^{1/2}$$

$$(*) \longrightarrow \quad \leq \left(2 \|(u_i v_i)_{i \in P}\|_1^2 \right)^{1/2}$$

$$= \sqrt{2} \, \|(u_i v_i)_{i \in P}\|_1$$

$$\alpha\beta \leq \tfrac{1}{4}(\alpha+\beta)^2 \longrightarrow \quad \leq \sqrt{2} \, \| (\tfrac{1}{4} (u_i + v_i)^2)_{i \in P} \|_1$$

$$= 2^{-\frac{3}{2}} \sum_{i \in P} (u_i + v_i)^2$$

$$\leq 2^{-\frac{3}{2}} \|u+v\|^2. \qquad \boxtimes$$

Lemma 3 Sei $(x_k, y_k, s_k) \in N(0,4)$. Dann gilt

$$\| X_{k+1} S_{k+1} e - \mu_{k+1} e \| \leq 0{,}2\, \mu_k.$$ D.h. insbesondere

$(x_{k+1}, y_{k+1}, s_{k+1}) \in N(0,4)$.

Bew.:

1. Beh.: $\| X_{k+1} S_{k+1} e - \mu_{k+1} e \| = \| \Delta X_k \Delta S_k e \|$.

Bew.: Betrachte i-ten Eintrag der Vektoren:

$$x_{k+1,i}\, s_{k+1,i} - \mu_{k+1} = (x_{k,i} + \Delta x_{k,i})(s_{k,i} + \Delta s_{k,i}) - \overset{L.1b)}{\widetilde{\sigma \mu_k}}$$

$$= x_{k,i} s_{k,i} + x_{k,i} \Delta s_{k,i} + \Delta x_{k,i} s_{k,i} + \Delta x_{k,i} \Delta s_{k,i} - \sigma \mu_k$$

$$= x_{k,i} s_{k,i} - x_{k,i} s_{k,i} + \sigma \mu_k + \Delta x_{k,i} \Delta s_{k,i} - \sigma \mu_k$$

$$= \Delta x_{k,i} \Delta s_{k,i}.$$

2. Beh.: $\| \Delta X_k \Delta S_k e \| \leq 0{,}2\, \mu_k$

Bew.: Definiere $D_k = X_k^{1/2} S_k^{-1/2}$. Dann

$$\| \Delta X_k \Delta S_k e \| = \| D_k^{-1} \Delta X_k D_k \Delta S_k e \|$$

$$\overset{L.2}{\leq} 2^{-\frac{3}{2}} \| D_k^{-1} \Delta x_k + D_k \Delta s_k \|^2.$$

Nun ist

$$D_k^{-1} \Delta x_k + D_k \Delta s_k = \left(X_k S_k\right)^{-1/2} \left(-X_k S_k e + \sigma \mu_k e\right),$$

weil

$$S_k \Delta x_k + X_k \Delta s_k = - X_k S_k e + \sigma \mu_k e$$

gilt, und dies multipliziert mit $(X_k S_k)^{-1/2}$ liefert die obige Gleichung.

Also

$$\| \Delta X_k \Delta S_k e \| \leq 2^{-\frac{3}{2}} \sum_{i=1}^{n} \frac{\left(-x_{k,i} s_{k,i} + \sigma \mu_k\right)^2}{x_{k,i} s_{k,i}}$$

$$\leq 2^{-\frac{3}{2}} \frac{\| X_k S_k e - \sigma \mu_k e \|^2}{\min\limits_i x_{k,i} s_{k,i}}$$

Da $(x_k, y_k, s_k) \in N(\sigma)$, gilt $\min x_{k,i} s_{k,i} \geq (1-\theta)\mu_k$,

(da $\| X_k S_k e - \mu_k e \| \leq \theta \mu_k \Rightarrow | x_{k,i} s_{k,i} - \mu_k | \leq \theta \mu_k$

$\Rightarrow x_{k,i} s_{k,i} - \mu_k \geq -\theta \mu_k$).

Desweiteren

$$e^T (X_k S_k - \mu_k e) = x_k^T s_k - \mu_k n = 0$$

Also

$$\| X_k S_k e - \sigma \mu_k e \|^2$$

$$= \| (X_k S_k e - \mu_k e) + (1-\sigma)\mu_k e \|^2$$

$$= \| X_k S_k e - \mu_k e \|^2 + 2(1-\sigma)\mu_k e^T (X_k S_k e - \mu_k e)$$

$$+ (1-\sigma)^2 \mu_k^2 e^T e.$$

$$\leq \theta^2 \mu_k^2 + (1-\sigma)^2 \mu_k^2 n.$$

Zusammen

$$\| \Delta X_k \Delta S_k e \| \leq 2^{-\frac{3}{2}} \frac{\theta^2 \mu_k^2 + (1-\sigma)^2 \mu_k^2 n}{(1-\theta)\mu_k}$$

$$\leq 0{,}2 \, \mu_k .$$

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<u>Literatur</u> zu diesem Kapitel:

S.J. Wright - Primal-dual interior-point methods,
SIAM, 1997.