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Methoden und Probleme der diskreten Mathematik

Wintersemester 2014/2015

— Aufgabenblatt 12 —

Aufgabe 12.1 Zeigen Sie, dass $r_3([m+n]) \leq r_3([m]) + r_3([n])$ gilt.

Aufgabe 12.2 Bestimmen Sie $r_3(15)$.

Aufgabe 12.3 Beweisen Sie das Theorem von Roth über arithmetische Progressionen in $\mathbb{Z}$, in dem Sie das Theorem von Roth über arithmetische Progressionen in $\mathbb{Z}/n\mathbb{Z}$ verwenden.

Aufgabe 12.4 Sei $\delta > 0$. Bestimmen Sie den Erwartungswert von

$$I_2 = \sum_{a \in \mathbb{Z}/n\mathbb{Z} \setminus \{0\}} \hat{1}_A(a)^2 \hat{1}_A(-2a),$$

wenn die Menge $A \subseteq \mathbb{Z}/n\mathbb{Z}$ auf die folgende Weise zufällig gewählt wurde: Ein Element $x \in \mathbb{Z}/n\mathbb{Z}$ gehört zu A mit Wahrscheinlichkeit δ.

Abgabe: Bearbeitete Aufgaben bis spätestens Mittwoch, den 21. Januar 2015 um 23 Uhr 59, in das Onlineformular auf der Vorlesungshomepage eintragen.

— Zitate —

Aus: W.T. Gowers — Rough structure and classification, 2000

2. Will Mathematics Exist in 2099?
[...]
Rather than giving several examples of the use of standard methods to solve problems, let me return to the question of automating mathematics and present an imagined dialogue between a mathematician and a computer in two or three decades' time. The idea of the dialogue is that the computer is very helpful to the mathematician, while not doing anything particular clever. This represents an unthreatening intermediate stage between what we have now, computer that act as slaves doing unbelievably boring calculations for us, and full automation of mathematics.
[...]
Mathematician. Is the following true? Let $\delta > 0$. Then for N sufficiently large, every set $A \subset \{1, 2, \dots, N\}$ of size at least δN contains a subset of the form $\{a, a+d, a+2d\}$?
Computer. Yes. If A is non-empty, choose $a \in A$ and set $d = 0$.
M. All right all right, but what if d is not allowed to be zero?
C. Have you tried induction on N, with some $\delta = \delta(N)$ tending to zero?
M. That idea is no help at all. Give me some examples please.
C. The obvious greedy algorithm gives the set

$$\{1, 2, 4, 5, 10, 11, 13, 14, 28, 29, 31, 32, 37, 38, 40, 41, \dots\}.$$

I notice that large parts of the set are translations of other part. In fact, this set is very like the Cantor set, so this gives a bound of $\delta \geq N^{(\log 2/\log 3)-1}$.
As for random methods, let us choose each element of $\{1, 2, \dots, N\}$ to be in A with probability δ, the choices being independent. The expected number of subsets of A of the form $\{a, a+d, a+2d\}$ is $\delta^3 N^2$, to within an absolute constant. Hence, if $\delta < N^{-2/3}$ it is possible for there to be none. The expected size of A is δN, and standard estimates tell us that A is very likely indeed to be of about this size.
Applying one further standard idea, we note that if the number of sets in A of the form $\{a, a+d, a+2d\}$ is at most half the number of points in A, then we can delete one element from each of them and still be left with an substantial proportion of A. This tells us that we can choose any δ that satisfied $C\delta^3 N^2 \leq \delta N$, so in particular we can find some δ of the form $cN^{-1/2}$.
M. Well, random methods often give the best answer for problems like this. Let's try to prove that $cN^{-1/2}$ is best possible.
C. [*Pauses for 0.001 seconds*] Actually it isn't. Behrend found a much better bound in 1946. [*Downloads paper*]
M. Oh dear, I'm out of ideas then. Could you give me a suggestion by any chance?
C. We have a set A. We want to prove that a subset of a certain form exists. The best way of proving existence is often to count.
M. [*Intrigued*] Yes, but what would that mean for a problem like this?
C. Here we wish to count the number of solutions (x, y, z) of the single linear equation $2y = x + z$. A standard tool in such situations is Fourier analysis, the circle method, whatever you like to call it.
M. Where is the group structure?
C. Identify $\{1, 2, \dots, N\}$ with $\mathbb{Z}/N\mathbb{Z}$. It isn't pretty but it sometimes works. I write $\mathbb{Z}_N$ for $\mathbb{Z}/N\mathbb{Z}$ and $\hat{A}(r)$ for $\sum_{s \in A} \exp(2\pi i r s/N)$. Then the number of triples $(a, a+d, a+2d) \in A^3$ is $N^{-1} \sum_{r \in \mathbb{Z}_N} \hat{A}(r)^2 \overline{\hat{A}(-2r)}$.
M. We are trying to show that that expression cannot be zero unless δ is very small.
C. [*Trained to humour mathematicians*] That is indeed almost equivalent to our original problem. Thank you. Notice that $N^{-1}\hat{A}(0)^3 = \delta^3 N^2$, exactly the number we had before when A was chosen randomly. This number is large and positive – a good sign.
M. So we would like the rest of the sum to be small enough not to cancel out the zero term.
C. Yes. I am trying a few obvious ideas such as the Cauchy-Schwarz inequality. Unfortunately, there does not seem to be any reason of the sum to be small.
M. Please show me your calculations.
C. Here they are. [*Displays them.*] They do enable me, or rather us, to prove a partial result. It states that if $\max_{r \neq 0} |\hat{A}(r)| \leq c\delta^2 N$, then the image of A in $\mathbb{Z}_N$ contains many sets of the desired form.
[...]