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Methoden und Probleme der diskreten Mathematik

Wintersemester 2014/2015

— Aufgabenblatt 13 —

Aufgabe 13.1 Bestimmen Sie den Wert von $\mathrm{ex}(n, \mathsf{K}_{1,r})$ für alle $n, r \in \mathbb{N}$.

Aufgabe 13.2 Sei $G = (V, E)$ ein Graph. Zeigen Sie, dass die Anzahl der Dreiecke in G wenigstens

$$\frac{4|E|}{3|V|}\left(|E| - \frac{|V|^2}{4}\right)$$

ist.

Hinweis: Jede Kante $\{u, v\} \in E$ ist in wenigstens $\deg(u) + \deg(v) - |V|$ Dreiecken enthalten.

Aufgabe 13.3 Sei $G = (V, E)$ ein Graph. Angenommen G enthält kein Dreieck. Zeigen Sie, dass

$$|E| \leq \alpha(G) \cdot \tau(G)$$

ist. Zur Erinnerung: $\tau(G)$ ist die minimale Kardinalität einer Knotenüberdeckung in G.

Hinweis: Es ist $\deg(v) \leq \alpha(G)$ für alle $v \in V$.

Aufgabe 13.4 Zeigen Sie, dass eine ε-reguläre Partition von G auch eine ε-reguläre Partition des Komplementärgraphen $\overline{G}$ ist.

Abgabe: Bearbeitete Aufgaben bis spätestens Mittwoch, den 28. Januar 2015 um 23 Uhr 59, in das Onlineformular auf der Vorlesungshomepage eintragen.

— Zitate —

Aus: M. Raussen, C. Skau — Interview with Endre Szemerédi, 2012

R&S: *We would now like to ask you some questions about your main contributions to mathematics. You have made some groundbreaking (we don't think that this adjective is an exaggeration) discoveries in combinatorics, graph theory, and combinatorial number theory. But arguably you are most famous for what is now called the Szemerédi theorem, the proof of the Erdős-Turán conjecture from 1936. Your proof is extremely complicated. The published proof is forty-seven pages long and has been called a masterpiece of combinatorial reasoning. Could you explain first of all what the theorem says, the history behind it, and why and when you got interested in it?*
Szemerédi: Yes, I will start in a minute to explain what it is, but I suspect that not too many people have read it. I will explain how I got to the problem, but first I want to tell how the whole story started. It started with the theorem of van der Waerden: you fix two numbers, say five and three. Then you consider the integers up to a very large number, from 1 to n, say. Then you partition this set into five classes, and there will always be a class containing a three-term arithmetic progression. That was a fundamental result of van der Waerden—of course, not only with three and five but with general parameters. Later, Erdős and Turán meditated over this result. They thought that maybe the reason why there is an arithmetic progression is not the partition itself; if you partition into five classes, then one class contains at least one fifth of all the numbers. They made the conjecture that what really counts is that you have dense enough sets. That was the Erdős-Turán conjecture: if your set is dense enough in the interval 1 to$n$—we are of course talking about integers—then it will contain a long arithmetic progression. Later Erdős formulated a very brave and much stronger conjecture: let's consider an infinite sequence of positive integers, $a_1 < a_2 < \ldots$, such that the sum of the inverses $1/a_i$ is divergent. Then the infinite sequence contains arbitrarily long arithmetic progressions. Of course, this would imply the absolutely fundamental result of Green and Tao about arbitrarily long arithmetic progressions within the primes because for the primes, we know that the sum of the inverses is divergent. That was a very brave conjecture; it isn't even solved for arithmetic progressions of length $k = 3$. But now people have come very close to proving it: Tom Sanders proved that if we have a subset between 1 and n containing at least n over $\log n(\log\log n)^5$ elements, then the subset contains a three-term arithmetic progression. Unfortunately, we need a little bit more, but we are getting close to solving Erdős's problem for $k = 3$ in the near future, which will be a great achievement. If I'm not mistaken, Erdős offered US $3,000 for the solution of the general case a long time ago. If you consider inflation, that means quite a lot of money.
[...]
R&S: *Let's get back to how you got interested in the problem.*
Szemerédi: That was very close to the Gelfand / Gelfond story, at least in a sense. At least the message is the same: I overlooked facts. I tried to prove that if you have an arithmetic progression, then it cannot happen that the squares are dense inside of it; specifically, it cannot be that a positive fraction of the elements of this arithmetic progression are squares. I was about twenty-five years old at the time and at the end of my university studies. At that time I already worked with Erdős. I very proudly showed him my proof, because I thought it was my first real result. Then he pointed out two, well, not errors, but deficiencies in my proof. Firstly, I had assumed that it was known that $r_4(n) = o(n)$, i.e., that if you have a set of positive upper density, then it has to contain an arithmetic progression of length four or, for that matter, of any length. I assumed that this was a true statement. Then I used [the fact] that there are no four squares that form an arithmetic progression. However, Erdős told me that the first statement was not known; it was an open problem. The other one was already known to Euler, which was two hundred fifty years before my time. So I had assumed something that is not known and, on the other hand, I had proved something that had been proven two hundred fifty years ago! The only way to try to correct something so embarrassing was to start working on the arithmetic progression problem. That was the time I started to work on $r_4(n)$ and, more generally, on $r_k(n)$. First I took a look at Klaus Roth's proof from 1953 of $r_3(n)$ being less than n divided by $\log\log n$. I came up with a very elementary proof for $r_3(n) = o(n)$ so that even high school students could understand it easily. That was the starting point. Later I proved also that $r_4(n) = o(n)$.