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BMS Summer School: Convex Geometry — Discrete and Computational

Packings, Coverings, and Embeddings

— Problem Sheet 1: Conic Optimization —

Problem 1.1

a) The set
$$\mathcal{D}^{n+1} = \{(X, s) \in \mathcal{S}^n_{\succeq 0} \times \mathbb{R}_{\geq 0} : (\det X)^{1/n} \geq s\}$$
is a proper convex cone.

b) Using the inequality of arithmetic and geometric means one can determine its dual cone $(\mathcal{D}^{n+1})^*$.

Problem 1.2 What is the smallest largest eigenvalue of a symmetric matrix of the form
$$X = \begin{pmatrix} 1 & X_{12} & 1 & 1 & X_{15} \\ X_{12} & 1 & X_{23} & 1 & 1 \\ 1 & X_{23} & 1 & X_{34} & 1 \\ 1 & 1 & X_{34} & 1 & X_{45} \\ X_{15} & 1 & 1 & X_{45} & 1 \end{pmatrix}?$$

Problem 1.3* The optimization problem over the completely positive cone $\mathcal{CP}_n$ and over the copositive cone $\mathcal{COP}_n$ are both NP-hard.

Problem 1.4* Let (X, d), with $X = \{x_1, \ldots, x_n\}$, be a finite metric space. The *minimal distortion* of (X, d) into Euclidean space is given by
$$c_2(X, d) = \min_{f : X \to \mathbb{R}^n \text{ injective}} \left(\max_{i \neq j} \frac{\|f(x_i) - f(x_j)\|}{d(x_i, x_j)} \cdot \max_{i \neq j} \frac{d(x_i, x_j)}{\|f(x_i) - f(x_j)\|} \right).$$
Then,
$$c_2(X, d) = \max_{Y \in \mathcal{S}^n_{\succeq 0}, Ye = 0} \sqrt{\frac{\sum_{ij : Y_{ij} > 0} Y_{ij} d(x_i, x_j)^2}{-\sum_{ij : Y_{ij} < 0} Y_{ij} d(x_i, x_j)^2}}.$$
The condition $Ye = 0$ says that the all-ones vector $e = (1, \ldots, 1)^{\mathsf{T}}$ lies in the kernel of Y.

Problem 1.5 What is the optimal distortion embedding of

a) the Petersen graph?

b) the skeleton of the permutahedron $\Pi_{n-1} = \operatorname{conv}\left\{(\sigma(1), \ldots, \sigma(n))^{\mathsf{T}} \in \mathbb{R}^n : \sigma \in S_n\right\}$?