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BMS Summer School: Convex Geometry — Discrete and Computational

Packings, Coverings, and Embeddings

— Problem Sheet 3: Coverings —

Problem 3.1 Given $Q \in \mathcal{S}^n$ and $V = \{v_1, \ldots, v_{n+1}\}$ be affinely independent. Let $c \in \mathbb{R}^n$ and $r > 0$ be such that $Q[c - v_i] = r^2$ for all $i = 1, \ldots, n+1$. Then,

$$\langle Q, N_{V,w} \rangle = Q[w - c] - r^2.$$

Problem 3.2 Consider $Q[x] = n \sum_{i=1}^n x_i^2 - \sum_{i \neq j} x_i x_j$. For a permutation $\pi \in S_{n+1}$ define the n-dimensional simplex

$$L_\pi = \mathrm{conv}\left\{ \sum_{i=1}^{k} e_{\pi(i)} : k = 1, \ldots, n+1 \right\},$$

where $e_{n+1} = -(e_1 + \cdots + e_n)$.

a) Then, $L_\pi \in \mathrm{Del}(Q)$ holds for any $\pi \in S_{n+1}$.

b) What is $\Delta(\mathrm{Del}(Q))$?

c) The lattice determined by Q is locally optimal for the lattice covering problem.

Problem 3.3 Define the *Cayley-Menger determinant* of n points $x_1, \ldots, x_n$, where the pairwise distances $d(x_i, x_j) = \|x_i - x_j\|$ are given, by

$$\mathrm{CM}(x_1, \ldots, x_n) = \begin{vmatrix} 0 & 1 & \ldots & 1 \\ 1 & d(x_1, x_1)^2 & \ldots & d(x_1, x_n)^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & d(x_n, x_1)^2 & \ldots & d(x_n, x_n)^2 \end{vmatrix}.$$

a) The Cayley-Menger determinant of $n + 2$ points in $\mathbb{R}^n$ vanishes.

b) Let $L = \mathrm{conv}\{v_0, \ldots, v_n\}$ be a n-dimensional simplex. Then the circumsphere of L has the squared radius

$$R^2 = -\frac{1}{2} \cdot \frac{\det\left(d(v_i, v_j)^2\right)_{0 \le i,j \le n}}{\mathrm{CM}(v_0, \ldots, v_n)}.$$

Problem 3.4* The Leech lattice Λ_{24} is locally optimal for the lattice covering problem.

Problem 3.5**

a) Is the E_8 root lattice locally optimal for the lattice packing-covering problem?

b) The Leech lattice Λ_{24} is optimal for the lattice packing-covering problem.