Convex Optimization (WS 2015/16)

Chapter I Introduction

mathematical program ($\hat{=}$ mathematical optimization problem)

given: functions $f_0, f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$,
constants $b_1, \ldots, b_m \in \mathbb{R}$

find: minimize $f_0(x)$
such that $x \in \mathbb{R}^n$
$$f_j(x) \le b_j \, , \quad j = 1, \ldots, m.$$

Vocabulary: f_0: objective function; $f_1, \ldots, f_m$: constraint functions; x: optimization variable; $x \in \mathbb{R}^n$ satisfying $f_j(x) \le b_j$ for all $j = 1, \ldots, m$: feasible solution; feasible solution x with $f_0(x) \le f_0(x')$ for all feasible solutions x': optimal solution

Important examples: Linear programs (LPs)
- all functions $f_0, \ldots, f_m$ are linear.

- LPs are well-understood in theory and practice ($\rightsquigarrow$ OR)

Nonlinear programs (NLPs)

One of the function $f_0, \dots, f_m$ is not linear:

$$f_0(x,y) = e^{\sin(50x)} + \sin(60e^y) + \sin(70\sin x)$$
$$+ \sin(\sin(80y)) - \sin(10(x+y)) + \frac{x^2+y^2}{4}$$

optimal solution $(-0{,}0244\dots, 0, 2106\dots)$

However, NLPs are too general for developing a useful mathematical theory.

Convex program

all function $f_0, \dots, f_m$ are convex.

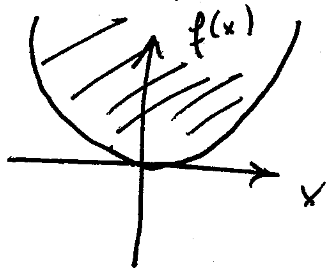
Recall:

Def. 1 A function $f: \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is called convex if the epigraph

$$\text{epi}(f) = \{(x, \alpha) \in \mathbb{R}^{n+1} : f(x) \le \alpha\}$$

is a convex set.

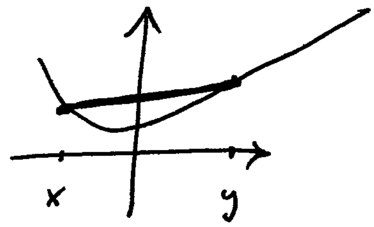
Ex. 2 $f(x) = x^2$ is convex

epi(f) = 

Proposition 3 A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex if and only if f satisfies <u>Jensen's inequality</u>:

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$$

for all $t \in [0,1]$, $x, y \in \mathbb{R}^n$.



Proof „$\Rightarrow$": Define $\tilde{x} = (x, f(x))$, $\tilde{y} = (y, f(y)) \in \mathbb{R}^{n+1}$.
Clearly, $\tilde{x}, \tilde{y} \in$ epi(f). Hence, $t\tilde{x} + (1-t)\tilde{y} \in$ epi(f) for $t \in [0,1]$. So,

$$(tx + (1-t)y,\ t f(x) + (1-t) f(y)) \in \text{epi}(f),$$

and

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y).$$

„$\Leftarrow$": very similar. ☒

Useful convexity criterion (OR 2015, Ex. 5.4)

<u>Proposition 4</u> Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function which is twice continuously differentiable. Function f is convex if and only if its Hessian

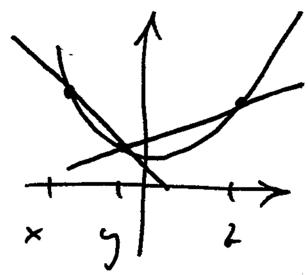
$$H(f)(z) = \left(\frac{\partial^2}{\partial x_i \partial x_j} f(z) \right)_{i,j=1,\dots,n} \in \mathbb{R}^{n \times n}$$

is positive semidefinite for all $z \in \mathbb{R}^n$.

[Schwarz's theorem implies that $H(f)$ is symmetric.].

<u>Proof</u> First consider the case $n = 1$. A function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if for all $x, y, z \in \mathbb{R}$ with $x < y < z$ inequality

$$(*) \qquad \frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(y)}{z - y}$$



holds. By the mean value theorem there are $\xi_1 \in [x, y]$, $\xi_2 \in [y, z]$ such that

$$f'(\xi_1) = \frac{f(y) - f(x)}{y - x} \quad \text{and} \quad f'(\xi_2) = \frac{f(z) - f(y)}{z - y}$$

hold. Suppose f is convex, then this together with $(*)$ implies that f' is monotonically increasing. Hence, $f'' \geq 0$.

Suppose $f'' \geq 0$. Then f' is monotonically increasing and $(*)$ is fulfilled. Hence, f is convex.

Now the multiple dimensional case: f is convex iff for all $x, y \in \mathbb{R}^n$ the function $g_{xy}(\alpha) = f(\alpha x + (1-\alpha) y)$ is convex. <u>To show:</u> $g''_{xy} \geq 0 \iff H_f(z)$ is positive semi-definite.

Apply chain rule to see this:

$$g'_{xy}(\alpha) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(y + \alpha(x-y))\,(x_i - y_i)$$

$$g''_{xy}(\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}\left(y + \alpha(x-y)\right)(x_i - y_i)(x_j - y_j).$$

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<u>fundamental properties of convex programs</u>

$$\begin{aligned} &\min f_0(x) \\ &x \in \mathbb{R}^n \\ &f_j(x) \leq b_j, \quad (f_0, \dots, f_m \text{ convex}). \end{aligned}$$

i) Set of feasible solutions is convex.
[immediately from Jensen's inequality].

ii) Local minima are global minima = optimal solution.

<u>Proposition 5</u> Consider the convex program

$$\min f_0(x)$$
$$x \in \mathbb{R}^n$$
$$f_j(x) \leq b_j, \quad j = 1, \dots, m,$$

where $C = \{x \in \mathbb{R}^n : f_j(x) \leq b_j, \ j = 1, \dots, m\}$ is the (convex) set of feasible solutions. Let x_0 be a <u>local minimizer</u> of the convex program, i.e. there is an $\varepsilon > 0$ s.t.

$$f_0(x_0) = \inf \{ f_0(y) : y \in C, \ \|x_0 - y\| \leq \varepsilon \}.$$

Then x_0 is an optimal solution of the convex program.

<u>Proof</u> Assume x_0 is not an optimal solution. Then there is a feasible solution $z \in C$ with $f(z) < f(x_0)$. Clearly, $\|x_0 - z\| > \varepsilon$. Define $y \in [x_0, z]$ by

$$y = (1-\alpha)x_0 + \alpha z \quad \text{with} \quad \alpha = \frac{\varepsilon}{\|x_0 - z\|} \in [0,1]$$

which lies in C. Then $\|x_0 - y\| = \varepsilon$ and by the convexity of f_0: $f(y) \leq (1-\alpha) f(x_0) + \alpha f(z) < f(x_0)$. ↯

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