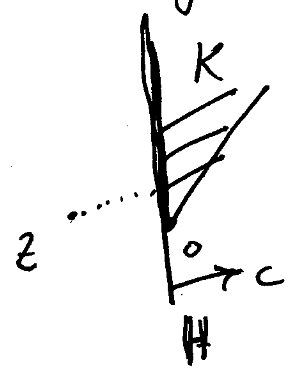
<u>Proof</u> a) is clear

b) recall from OR : separation theorem

Let $K \subseteq \mathbb{R}^n$ be a convex cone which is closed.
Suppose $z \notin K$. Then there is a $c \in \mathbb{R}^n$ so
that $c^T z < 0$ and $c^T x \geq 0$ for all $x \in K$.

Hence, the hyperplane $H = \{y \in \mathbb{R}^n : c^T y = 0\}$ separates $\{z\}$ and K strictly.



To show: $K \supseteq (K^*)^*$.

Suppose $z \in \mathbb{R}^n \setminus K$. Then there is $c \in \mathbb{R}^n$ with $c^T z < 0$ and $c^T x \geq 0$ for all $x \in K$. Hence, $c \in K^*$ and $z \notin (K^*)^*$. $\boxtimes$

Back to the examples:

a) $(\mathbb{R}^n_{\geq 0})^* = \mathbb{R}^n_{\geq 0}$ $\rightsquigarrow$ clear

b) $(\mathcal{L}^{n+1})^* = \mathcal{L}^{n+1}$ $\rightsquigarrow$ exercise

c) $(\mathcal{S}^m_{\geq 0})^* = \mathcal{S}^m_{\geq 0}$. $\rightsquigarrow$ needs some discussion.

§ 2 The PSD cone

$S^n = \{ X \in \mathbb{R}^{n \times n} : X \text{ symmetric}, X_{ij} = X_{ji} \}$
is the vector space of symmetric matrices. It has dimension $\binom{n+1}{2} = \frac{n(n+1)}{2}$. It is a Euclidean vector space with inner product

$$\langle X, Y \rangle = \mathrm{Tr}(Y^T X) = \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij} Y_{ij},$$

where $\mathrm{Tr}(X) = \sum_{i=1}^{n} X_{ii}$, the <u>trace</u> of matrix X.

<u>Lemma 1</u> $(S^n, \langle \cdot, \cdot \rangle)$ is isometric to
$\left(\mathbb{R}^{\frac{n(n+1)}{2}}, (x,y) \mapsto x^T y\right)$.

<u>Proof</u> The following linear map provides the isometry

$$T : S^n \to \mathbb{R}^{\frac{n(n+1)}{2}}$$

$$X \mapsto (X_{11}, \sqrt{2} X_{12}, \sqrt{2} X_{13}, \dots, \sqrt{2} X_{1n},$$
$$X_{22}, \sqrt{2} X_{23}, \dots, \sqrt{2} X_{2n},$$
$$\ddots$$
$$X_{nn})$$

take upper triangular matrix of X; scale non-diagonal entries.

Proposition 2 „spectral decomposition"

Every symmetric $X \in S^n$ has a spectral decomposition, i.e. there is an orthonormal basis $u_1, \dots, u_n \in \mathbb{R}^n$ and real numbers $\lambda_1, \dots, \lambda_n$ so that

$$X = \sum_{i=1}^{n} \lambda_i u_i u_i^T$$

holds. [$\lambda_1, \dots, \lambda_n$ are eigenvalues of X and $u_1, \dots, u_n$ corresponding eigenvectors.]

in matrix form: $X = PDP^T$, where $P \in O(n)$ is a matrix in the orthogonal group $O(n)$ [$PP^T = P^TP = I$ = identity matrix.] and where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is a diagonal matrix.

Proof $\rightarrow$ Lineare Algebra.

Def. 3 a) $X \in S^n$ is called positive semidefinite if for all $v \in \mathbb{R}^n$: $v^T X v \geq 0$ holds.

b) $X \in S^n$ is positive definite if for all $v \in \mathbb{R}^n \setminus \{0\}$: $v^T X v > 0$.

Notation: $X \succeq 0$; $X \succ 0$

$\uparrow$ \$\succeq\$

Note that $v^T X v = \langle X, vv^T \rangle$ holds.

<u>Proposition 4</u> For $X \in S^n$ the following statements are equivalent:

(a) X is positive semidefinite

(b) smallest eigenvalue of X is nonnegative

(c) $\exists L \in \mathbb{R}^{n \times k}$ with $X = LL^T$, a <u>Cholesky factorisation</u> of X.

(d) $\exists v_1, \dots, v_n \in \mathbb{R}^k$ with $X_{ij} = v_i^T v_j$, a Gram representation of X.

(e) all principal minors of X are nonnegative.

<u>Proof</u> <u>(a) $\Rightarrow$ (b)</u>: Spectral decomposition of X:

$X = \sum_{i=1}^{n} \lambda_i u_i u_i^T$. For $j \in \{1, \dots, n\}$ we have

$$0 \leq u_j^T X u_j = u_j^T \sum_{i=1}^{n} \lambda_i u_i \underbrace{(u_i^T u_j)}_{= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}} = \lambda_j u_j^T u_j = \lambda_j .$$

<u>(b) $\Rightarrow$ (a)</u>: For $v \in \mathbb{R}^n$ we have

$$v^T X v = v^T \sum_{i=1}^{n} \lambda_i u_i u_i^T v = \sum_{i=1}^{n} \underbrace{\lambda_i}_{\geq 0} \underbrace{(u_i^T v)^2}_{\geq 0} \geq 0 .$$

other implications: $\rightsquigarrow$ Lineare Algebra.

Addition to Proposition 4 in the case X is positive definite:

b) smallest eigenvalue of X is strictly positive

c) $\exists\, L \in \mathbb{R}^{n \times n}$ with rank $L = n$ and $X = LL^T$.

d) $v_1, \dots, v_n$ are linearly independent.

e) all leading principal minors are positive.

<u>Def. 5</u> $\mathcal{S}^n_{\succeq 0} = \{ X \in \mathcal{S}^n : X \text{ is positive semidefinite} \}$ is the cone of <u>positive</u> <u>semidefinite</u> <u>matrices</u> (<u>PSD cone</u>),

$\mathcal{S}^n_{\succ 0} = \{ X \in \mathcal{S}^n : X \text{ positive definite} \}$.

<u>Proposition 6</u> $\mathcal{S}^n_{\succeq 0}$ is a proper convex cone. Its interior is $\mathcal{S}^n_{\succ 0}$.

<u>Proof</u>
- $\mathcal{S}^n_{\succeq 0}$ is convex cone: easy calculation
- $\mathcal{S}^n_{\succeq 0}$ is closed: by definition ($\mathcal{S}^n_{\succeq 0}$ is intersection of infinitely many closed halfspaces)
- $\mathcal{S}^n_{\succeq 0}$ is pointed: suppose $X, -X \in \mathcal{S}^n_{\succeq 0}$. The smallest eigenvalue of $-X$ is nonnegative. Denote it by $\lambda_{min}(-X)$. We have $\lambda_{min}(-X) = -\lambda_{max}(X)$ and so $-\lambda_{max}(X) \geq 0$. Hence, $\lambda_{max}(X) = 0$, and $X = 0$ follows.
- $\operatorname{int} \mathcal{S}^n_{\succeq 0} = \mathcal{S}^n_{\succ 0}$: $\longrightarrow$ exercise. $\boxtimes$