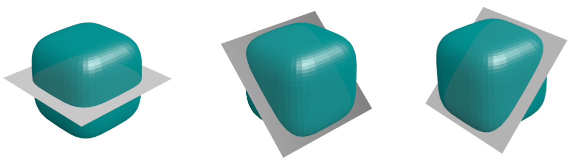
SOS polynomials invariant under finite reflection groups

Maria Dostert (University of Cologne) Cristóbal Guzmán (Georgia Tech) Fernando M. de Oliveira (University of São Paulo) Frank Vallentin (University of Cologne)

Finite reflection group B_3

generated by three reflecting planes:



B_3 = symmetry group of the regular cube $[-1,+1]^3$
= symmetry group of supersphere B_3^p where
$B_3^p = \{x \in \mathbb{R}^3 : |x_1|^p + |x_2|^p + |x_3|^p \leq 1\},\ p \geq 1$

B_3 is acting on $\mathbb{C}[x_1,x_2,x_3] = \mathbb{C}[x]$ by $(gp)(x) = p(g^{-1}x)$

invariant ring $\mathbb{C}[x]^{\mathsf{B}_3} = \{p \in \mathbb{C}[x] : gp = p\ \forall g \in \mathsf{B}_3\}$
$= \mathbb{C}[\theta_1,\theta_2,\theta_3]$ with $\theta_i = x_1^{2i} + x_2^{2i} + x_3^{2i}$,

where $\theta_1,\theta_2,\theta_3$ are basic invariants which are algebraically independent homogeneous polynomials. (Chevalley-Shephard-Todd)

SOS polynomials

$p \in \mathbb{R}[x]_{2d}$ is a sum of squares (SOS). $\Longleftrightarrow$ There is a positive semidefinite matrix $Q \in \mathbb{R}^{\binom{n+d}{d} \times \binom{n+d}{d}}$ such that $p(x) = b(x)^{\mathsf{T}}\, Q\, b(x)$,

where $b(x) \in \mathbb{R}[x]^{\binom{n+d}{d}}$ contains a basis of the space of polynomials up to degree d.

Theorem (Gatermann, Parrilo 2002)

Let G be a finite group generated by reflections.
The cone of G-invariant polynomials which can be written as a SOS is equal to

$$\{p \in \mathbb{R}[x] : p = \sum_{\pi \in \widehat{G}} \langle P^\pi, Q^\pi \rangle,\ P^\pi \text{ is Hermitian SOS matrix polynomial in } \theta_1,\ldots,\theta_n\},$$

where $Q^\pi \in (\mathbb{C}[x]^G)^{d_\pi \times d_\pi}$ is defined by $[Q^\pi]_{kl} = \sum_{i=1}^{d_\pi} \varphi^\pi_{ki} \overline{\varphi^\pi_{li}}$.

φ^π is a special basis of the coinvariant algebra (described below)

computational advantage: smaller block matrices

For $n = 3, d = 15$: $Q \in \mathcal{S}^{816}_{\succeq 0}$ vs. $Q^{\pi_i} \in \mathcal{S}^{d_i}_{\succeq 0}$ for

i	1	2	3	4	5	6	7	8	9	10
d_i	31	23	11	7	27	39	34	50	50	70

Coinvariant algebra $\mathbb{C}[x]_G$

$\mathbb{C}[x]_G$ is a $|G|$-dimensional graded algebra defined by

$$\mathbb{C}[x]_G = \mathbb{C}[x]/(\theta_1,\ldots,\theta_n) = \oplus_{k=0}^m (\mathbb{C}[x]_G)_k$$

where m = maximal degree of the Poincáre polynomial. In particular, we have

$$\mathbb{C}[x] = \mathbb{C}[x]^G \otimes \mathbb{C}[x]_G$$

$\widehat{G}$ = set of irreducible unitary representations of G up to equivalence.

$\mathbb{C}[x]_G$ equivalent to the regular representation of G: There are homogeneous polynomials

$$\varphi^\pi_{ij},\ \text{with } \pi \in \widehat{G}, 1 \leq i,j \leq d_\pi,$$

where d_π is the degree of π. They form a basis of $\mathbb{C}[x]_G$ such that

$$g\varphi^\pi_{ij} = (\pi(g)_{1j} \ldots \pi(g)_{d_\pi j}) \begin{pmatrix} \varphi^\pi_{i1} \\ \vdots \\ \varphi^\pi_{id_\pi} \end{pmatrix},\ i = 1,\ldots,d_\pi.$$

Calculation of φ^π_{ki}

finite group G generated by reflections $s_1,\ldots,s_n$

$$\Big\downarrow \text{primary invariants } \theta_1,\ldots,\theta_n$$

$$V = \mathbb{C}[x]/(\theta_1,\ldots,\theta_n)$$

$$\Big\downarrow \text{representation } \rho_k : G \to GL,\ \rho_k(g)(p) \mapsto gp;\quad p^\pi_k : V_k \to V^\pi_k,\quad p^\pi_k = \frac{d_\pi}{|G|}\sum_{g \in G}\sum_{i=1}^{d_\pi} \pi_{ii}(g^{-1})\rho_k(g)$$

$$V = \bigoplus_{\pi \in \widehat{G}} \bigoplus_k V^\pi_k$$

$$\Big\downarrow p^\pi_{k,ii} : V^\pi_k \to V^\pi_{k,i},\quad p^\pi_{k,ij} = \frac{d_\pi}{|G|}\sum_{g \in G} \pi_{ji}(g^{-1})\rho_k(g)$$

$$V^\pi_k = \bigoplus_{i=1}^{d_\pi} V^\pi_{k,i}$$

$$\Big\downarrow \text{consider a non-zero vector } \varphi^\pi_{k1} \in V^\pi_{k,1}$$

$$\varphi^\pi_{ki} = p^\pi_{k,i1}(\varphi^\pi_{k1})$$

New upper bounds for the density of translative packings of superspheres

Theorem (Cohn, Elkies 2003)

Suppose f satisfies the following conditions:

(i) $\widehat{f}(0) \geq \operatorname{vol} B_3^p$,
(ii) $\widehat{f}(u) \geq 0$ for every $u \in \mathbb{R}^3 \setminus \{0\}$,
(iii) $f(x) \leq 0$ whenever $(B_3^p)^\circ \cap (x + (B_3^p)^\circ) = \emptyset$.

Then the density of any packing of translates of B_3^p in $\mathbb{R}^3$ is at most $f(0)$.

Specify a function $f : \mathbb{R}^3 \to \mathbb{R}$ via its Fourier transform $\widehat{f}$:

$$\widehat{f}(u) = p(u)e^{-\pi\|u\|^2}, \text{ where } p \text{ is a polynomial.}$$

We can assume: f is B_3-invariant. Thus p is also B_3-invariant.

If f satisfies (i) – (iii), then so does $Mf(x) = \frac{1}{|\mathsf{B}_3|}\sum_{g \in \mathsf{B}_3} f(g^{-1}x)$, which is B_3-invariant.

For (i), check whether $p(0) \geq \operatorname{vol} B_3^p$.

For (ii), check whether p is SOS (using Gatermann-Parrilo).

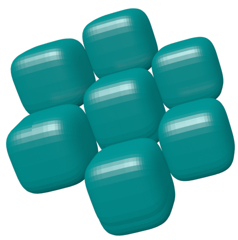
For (iii), 1. compute f from $\widehat{f}$ by a linear transformation $f(x) = \int_{\mathbb{R}^3} \widehat{f}(u)e^{2\pi i u \cdot x}\mathrm{d}u$,
2. check constraint via SOS conditions (using Gatermann-Parrilo).

Thus we find such a function f by solving a finite semidefinite program.

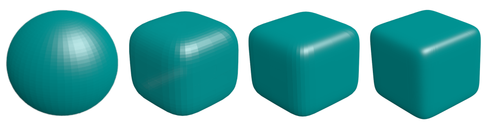
Computational results

Lower bounds: lattice packings of Jiao, Stillinger, and Torquato (2009)

lattice packing for $p = 4$: every supersphere has 12 neighbours



Upper bounds: $p = 2$: coincides with upper bound of Cohn & Elkies
$p = 4,\ 6,\ 8$: new upper bounds



p	2	4	6	8
lower bound	0.7404	0.8698	0.9318	0.9582
upper bound	0.7797	0.8731	0.9331	0.9594