Proposition 7 $(S^n_{\geq 0})^* = S^n_{\geq 0}$; i.e. the PSD cone is self-dual.

Recall (Lemma 1): $(S^n_{\geq 0})^* = \{ Y \in S^n : \langle X, Y \rangle \geq 0$ for all $X \in S^n_{\geq 0} \}$.

Proof: $\underline{S^n_{\geq 0} \subseteq (S^n_{\geq 0})^*}$: Let $X \in S^n_{\geq 0}$ be given. For $Y \in S^n_{\geq 0}$ consider the spectral decomposition $Y = \sum_{i=1}^n \lambda_i u_i u_i^T$ where $\lambda_i \geq 0$ (Proposition 4(b)). Then,

$$\langle X, Y \rangle = \langle X, \sum \lambda_i u_i u_i^T \rangle = \sum \lambda_i \langle X, u_i u_i^T \rangle$$
$$= \sum \underbrace{\lambda_i}_{\geq 0} \underbrace{u_i^T X u_i}_{\geq 0} \geq 0.$$

Hence, $X \in (S^n_{\geq 0})^*$.

$\underline{S^n_{\geq 0} \supseteq (S^n_{\geq 0})^*}$: For $x \in \mathbb{R}^n$ the rank-1-matrix $xx^T \in S^n_{\geq 0}$.

Consider $Y \in (S^n_{\geq 0})^*$. Then,

$$0 \leq \langle xx^T, Y \rangle = x^T Y x \Rightarrow Y \in S^n_{\geq 0}. \qquad \boxtimes$$

In particular: If $X \in S^n$ satisfies all linear inequalities $\langle X, Y \rangle \geq 0$, with $Y \in S^n_{\geq 0}$, then $X \in S^n_{\geq 0}$.

Proposition 8 (Schur complement)

Let $X \in S^n$ be a matrix in block diagonal form

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}, \quad \text{with } A \in S^p, \ C \in S^{n-p}, \ B \in \mathbb{R}^{p \times (n-p)}.$$

Suppose A is invertible. Then $X \in S^n_{\geq 0}$ if and only if $A \in S^p_{\geq 0}$ and $C - B^T A^{-1} B \in S^{n-p}_{\geq 0}$.

The matrix $C - B^T A^{-1} B$ is called the Schur complement of A in X.

Proof direct check:

$$X = P^T \begin{bmatrix} A & 0 \\ 0 & C - B^T A^{-1} B \end{bmatrix} P, \quad \text{with } P = \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}. \quad \boxtimes$$

Def. 9 Define the Schur - Hadamard product of $X, Y \in S^n$ by $(X \circ Y)_{ij} = X_{ij} Y_{ij}$.

Proposition 10 Suppose $X, Y \in S^m_{\geq 0}$, then $X \circ Y \in S^m_{\geq 0}$.

Proof direct verification using spectral decomposition.

Def. 11 The Kronecker product of $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$ is $A \otimes B \in \mathbb{R}^{mp \times nq}$ defined by

$$(A \otimes B)_{(ih, jk)} = A_{ij} B_{hk}.$$

$A \otimes B$ is an $m \times n$ block matrix with blocks of size $p \times q$ (and vice versa):

$$A \otimes B = \begin{bmatrix} A_{11} B & A_{12} B & \dots & A_{1n} B \\ & \dots & & \\ A_{m1} B & A_{m2} B & \dots & A_{mn} B \end{bmatrix}.$$

Proposition 12 a) $(A \otimes B)(C \otimes D) = AC \otimes BD$

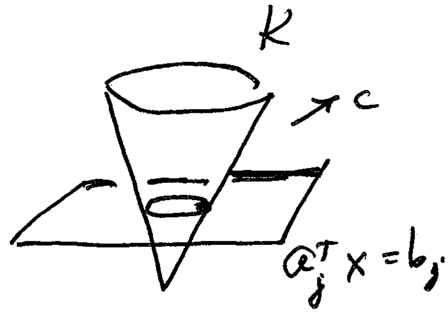
b) Assume $A \in S^m$, $B \in S^n$ have eigenvalues $\alpha_1, \dots, \alpha_m$, resp. $\beta_1, \dots, \beta_n$. Then $A \otimes B$ has eigenvalues $\alpha_i \beta_j$ with $i \in [m] = \{1, \dots, m\}$, $j \in [n]$.

c) If A, B are positive semidefinite, then $A \otimes B$ is positive semidefinite too.

Proof direct verification.

§ 3 Conic programs

<u>Def. 1</u> Let $K \subseteq \mathbb{R}^n$ be a proper convex cone. Given $c, a_1, \dots, a_m \in \mathbb{R}^n$, $b_1, \dots, b_m \in \mathbb{R}$. A <u>primal conic program</u> (in standard form) is the following maximisation problem



$$
(P) \quad p^* = \sup \; c^T x \quad x \in \mathbb{R}^n, \quad a_j^T x = b_j, \; j \in [m], \quad x \in K
$$

The corresponding <u>dual conic program</u> is the following minimisation problem

$$
b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}
$$

$$
(D) \quad d^* = \inf \; b^T y \quad y \in \mathbb{R}^m, \quad \sum_{j=1}^m y_j a_j - c \in K^*
$$

<u>Remark</u> (a) We often write $Ax = b$ with $A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} \in \mathbb{R}^{m \times n}$

instead of $a_j^T x = b_j$.

Similarly: $A^T y - c$ instead of $\sum_{j=1}^m y_j a_j - c$.

(b) $x \in \mathbb{R}^n$ is <u>feasible</u> for (P) if $Ax = b$ and $x \in K$.
x is <u>strictly feasible</u> if x feasible and $x \in \operatorname{int} K$.

Similarly, y is feasible for (D) if $A^T y - c \in K^*$, and strictly feasible if $A^T y - c \in \operatorname{int} K^*$.

Geometric interpretation of (P) and (D)

Define subspaces

$$L = \{ x \in \mathbb{R}^n : Ax = 0 \}$$

$$L^\perp = \{ y \in \mathbb{R}^n : x^T y = 0 \text{ for all } x \in L \}$$

$$= \{ y^T A : y \in \mathbb{R}^m \}.$$

Suppose there exists $x_0 \in \mathbb{R}^n$ with $A x_0 = b$; otherwise (P) does not have feasible solution. Then,

$$b^T y = (A x_0)^T y = x_0^T A^T y = x_0^T (A^T y - c) + x_0^T c,$$

and

$$p^* = \sup \{ c^T x : x \in K \cap (x_0 + L) \}$$

$$d^* = c^T x_0 + \inf \{ x_0^T z : z \in K^* \cap (-c + L^\perp) \}.$$

Rewrite the last equation in primal form

$$d^* = c^T x_0 - \sup \{ -x_0^T z : z \in K^* \cap (-c + L^\perp) \}$$

and take the dual. Then one gets back the primal because $(K^*)^* = K$ holds by Proposition 1.4.