Examples

Def. 2 $K_1 \subseteq \mathbb{R}^m$, $K_2 \subseteq \mathbb{R}^n$ convex cones. Direct product

$$K_1 \times K_2 = \{(x,y) \in \mathbb{R}^{m \times n} : x \in K_1, y \in K_2\}.$$

Proposition 3 a) If K_1, K_2 are both proper convex cones, then so is $K_1 \times K_2$.

b) $(K_1 \times K_2)^* = K_1^* \times K_2^*$.

Proof easy verification.

(A) Linear programming (LP)

$K = \mathbb{R}^n_{\geq 0}$.

$$p^* = \sup\{c^T x : x \in \mathbb{R}^n, Ax = b, \overbrace{x \in \mathbb{R}^n_{\geq 0}}^{x \geq 0 \text{ component-wise}}\}$$

$$d^* = \inf\{b^T y : A^T y - c \in \mathbb{R}^n_{\geq 0}\}.$$

(B) Conic quadratic programming (CQP)

$K = \mathcal{L}^{n_1+1} \times \mathcal{L}^{n_2+1} \times \cdots \times \mathcal{L}^{n_r+1}$

$$p^* = \sup \; ((c_1,\gamma_1),\dots,(c_r,\gamma_r))^T((x_1,t_1),\dots,(x_r,t_r))$$

$$((x_1,t_1),\dots,(x_r,t_r)) \in \mathcal{L}^{n_1+1} \times \cdots \times \mathcal{L}^{n_r+1}$$

$$((a_{j1},\alpha_{j1}),\dots,(a_{jr},\alpha_{jr}))^T((x_1,t_1),\dots,(x_r,t_r)) = b_j, \quad j \in [m].$$

$$d^* = \inf b^T y$$
$$y \in \mathbb{R}^m$$
$$(*) \qquad \sum_{j=1}^m y_j \left((a_{j1}, \alpha_{j1}), \dots, (a_{jr}, \alpha_{jr}) \right) - \left((c_1, \gamma_1), \dots, (c_r, \gamma_r) \right) \in \mathcal{L}^{n_1+1} \times \dots \times \mathcal{L}^{n_r+1}$$

More intuitively: Define matrices

$$A_i = \left[a_{1i} \cdots a_{mi} \right] \in \mathbb{R}^{n_i \times m}, \quad i = 1, \dots, r,$$

and vectors $d_i = (\alpha_{1i}, \dots, \alpha_{mi})^T$, $i = 1, \dots, r$.

Then, $(*) \iff \| A_i y - c_i \| \leq d_i^T y - \gamma_i$, $i = 1, \dots, r$.

In particular, one recognises that LP is a special case of CQP: set $A_i = 0$, $c_i = 0$.

(c) Semidefinite programming (SDP)

$K = S^n_{\succeq 0}$. $\qquad p^* = \sup \langle C, X \rangle$
$$X \in S^n_{\succeq 0}$$
$$\langle A_j, X \rangle = b_j, \quad j \in [m].$$

$$d^* = \inf b^T y$$
$$y \in \mathbb{R}^m$$
$$\sum_{j=1}^m y_j A_j - C \in S^n_{\succeq 0}$$

Inequalities of the form $\sum_{j=1}^{m} y_j A_j - C \succeq 0$ are called
<u>linear matrix inequalities</u> (LMIs)

- LP is a special case of SDP: X is diagonal matrix
- CQP is a special case of SDP:

<u>Lemma 4</u> $\mathcal{L}^{n+1} = \{ (x,t) \in \mathbb{R}^{n+1} : \|x\| \le t \}$
$= \{ (x,t) \in \mathbb{R}^{n+1} : \begin{bmatrix} tI_n & x \\ x^T & t \end{bmatrix} \succeq 0 \}$.

<u>Proof</u> If $t = 0$, then $x = 0$.
If $t > 0$, then $t I_n \succ 0$.
Consider the Schur complement of $t I_n$ in $\begin{bmatrix} tI_n & x \\ x^T & t \end{bmatrix}$:
Using Proposition 2.8: $\begin{bmatrix} tI_n & x \\ x^T & t \end{bmatrix}$ is positive semidefinite
if and only if $t - x^T \frac{1}{t} I_n x \geq 0 \iff t^2 \geq x^T x$
$\iff t \geq \|x\|$. $\boxtimes$

§4 Theorem of alternatives

Recall: Farkas Lemma from OR.

Lemma 1 For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ exactly one of the following two alternatives holds:

(i) $\exists\, x \geq 0 : \; Ax = b$

(ii) $\exists\, y \in \mathbb{R}^m : \; A^T y \geq 0 \,, \; b^T y < 0.$

Goal: Generalise to conic programs; replace "≥ 0" by cone condition.

First try: Let $K \subseteq \mathbb{R}^n$ be a proper convex cone. For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ exactly one of the following two alternatives holds:

(i) $\exists\, x \in K : \; Ax = b$

(ii) $\exists\, y \in \mathbb{R}^m : \; A^T y \in K^*, \; b^T y < 0.$

"Proof" (i) $\Rightarrow \neg$ (ii) Suppose x is feasible for (i) and y is feasible for (ii). Then,

$$0 \leq (A^T y)^T x = y^T A x = y^T b < 0. \quad \text{Contradiction!}$$