$\neg$(i) $\Rightarrow$ (ii) Suppose (i) has no solution. Then, $b \notin \{Ax : x \in K\}$, where $\{Ax : x \in K\}$ is a convex cone. By the separation theorem there is a vector $y \in \mathbb{R}^m$ with $y^T b < 0$ and $y^T A x \geq 0$ for all $x \in K$. Hence, $(y^T A)^T = A^T y \in K^*$ and (ii) follows.

Example 2 Define $E_{ij} = \frac{1}{2}(e_i e_j^T + e_j e_i^T)$, $1 \leq i \leq j \leq n$.

Consider (i) $\langle E_{11}, X \rangle = 0$, $\langle E_{12}, X \rangle = 1$, $X \in S^2_{\succeq 0}$.

(ii) $y_1 E_{11} + y_2 E_{12} \succeq 0$, $y_2 < 0$.

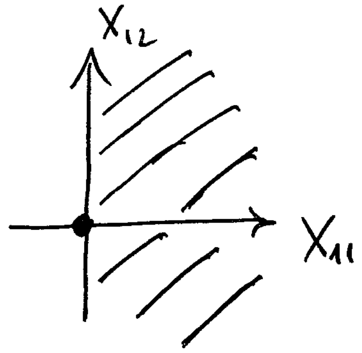
Neither (i) nor (ii) has a feasible solution!

Where is the problem?

The convex cone

$$A S^2_{\succeq 0} = \left\{ \begin{bmatrix} \langle E_{11}, X \rangle \\ \langle E_{12}, X \rangle \end{bmatrix} : X \in S^2_{\succeq 0} \right\}$$
$$= \left\{ \begin{bmatrix} X_{11} \\ X_{12} \end{bmatrix} : X_{11} > 0, X_{12} \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

is not closed.



Def. 2 Let $A \in \mathbb{R}^{m \times n}$ be a matrix and let $b \in \mathbb{R}^m$ be a vector. The system

$$Ax = b, \quad x \in K$$

is called <u>weakly feasible</u> if for all $\varepsilon > 0$ there exists $x \in K$ with $\|Ax - b\| \leq \varepsilon$

In other words, there is a sequence $(x_k)_{k \in \mathbb{N}}$ with $x_k \in K$ and $\lim_{k \to \infty} Ax_k = b$. [weakly feasible $\hat{=}$ <u>limit-feasible</u>].

Theorem 3 For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ exactly one of the following two alternatives holds true:

i) The system $Ax = b$, $x \in K$ is weakly feasible

ii) There exists $y \in \mathbb{R}^m$ with $A^T y \in K^*$, $b^T y < 0$.

Proof Same argument as <u>above</u>, but choose the topological closure $\{Ax : x \in K\}$.

The system of the example

$$\langle E_{11}, X \rangle = 0, \quad \langle E_{12}, X \rangle = 1, \quad X \succeq 0$$

is weakly feasible: Choose the sequence $X_k = \begin{bmatrix} \frac{1}{k} & 1 \\ 1 & k \end{bmatrix}$, $k \in \mathbb{N}$.

<u>Corollary 4</u> For $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$ exactly one of the following two alternatives holds true:

i) $\exists x \in K$: $Ax = 0$ and $c^T x > 0$

ii) The system $A^T y - c \in K^*$ is <u>weakly feasible</u>, i.e. $\forall \varepsilon > 0\ \exists y \in \mathbb{R}^m$ and $z \in K^*$ such that $\| A^T y - c - z \| \leq \varepsilon$.

<u>Proof</u> $\rightarrow$ Exercise

<u>Theorem 5</u> Suppose that the linear system $Ax = b$ has a solution x_0, then exactly one of the following two alternatives holds true:

i) $\exists x \in \operatorname{int} K$: $Ax = b$ (x is <u>strictly feasible</u>)

ii) $\exists y \in \mathbb{R}^m$: $A^T y \in K^* \setminus \{0\}$ and $b^T y \leq 0$.

In the running example $\langle E_{11}, X \rangle = 0$, $\langle E_{12}, X \rangle = 1$, $X \succeq 0$

condition ii) is fulfilled:

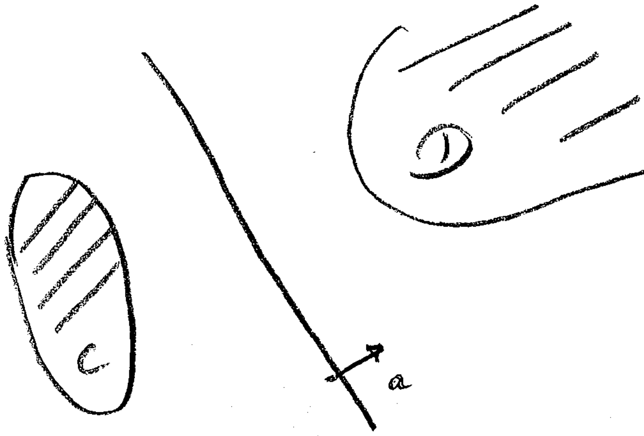
$$y_1 E_{11} + y_2 E_{12} \succeq 0 \qquad \text{with } y_1 = 1,\ y_2 = 0.$$
$$y_2 \leq 0$$

For the proof we need the following variant of the separation theorem.

<u>Theorem 6</u> Let $C, D \subseteq \mathbb{R}^n$ be nonemptly convex sets which do not intersect. Then there is a vector $a \in \mathbb{R}^n \setminus \{0\}$ such that

$$\sup_{x \in C} a^T x \leq \inf_{x \in D} a^T x$$

holds.



<u>Proof</u> $\rightarrow$ OR (2015)

<u>Proof</u> (of Thm. 5)

<u>i) $\Rightarrow$ ¬ii)</u> Suppose $\exists x \in \text{int } K$ with $Ax = b$ and suppose $\exists y \in \mathbb{R}^m : A^T y \in K^* \setminus \{0\}$ with $b^T y \leq 0$.

Then,

$$0 \leq (A^T y)^T x = y^T A x = y^T b \leq 0 .$$

Hence, $(A^T y)^T x = 0$ in contradiction to Exercise 2.4.

$$[x \in \text{int } K \Leftrightarrow y^T x > 0 \quad \forall y \in K^* \setminus \{0\}].$$

<u>¬i) $\Rightarrow$ ii)</u> Suppose $Ax = b$ does not have a solution $x \in \text{int } K$. Consider the linear subspace $L = \{x : Ax = 0\}$.

We have

$$x_0 + L \cap \operatorname{int} K = \emptyset.$$

Using Theorem 6 gives a vector $a \in \mathbb{R}^n \setminus \{0\}$ and an $\alpha \in \mathbb{R}$ satisfying

$$a^T x \geq \alpha \quad \forall x \in K \quad \text{and} \quad a^T x \leq \alpha \quad \forall x \in x_0 + L.$$

Since $0 \in K$, the inequality $0 \geq \alpha$ follows.

Furthermore, $a \in K^*$, since $a^T(tx) \geq \alpha \quad \forall t > 0,\ x \in K$.

Also $a \in L^\perp$, since $a^T(tx + x_0) \leq \alpha \quad \forall t \in \mathbb{R},\ x \in L$.

There exists $y \in \mathbb{R}^m$ with $a = A^T y$ because $L^\perp = \{A^T y : y \in \mathbb{R}^m\}$.

Together, $A^T y \in K^* \setminus \{0\}$ and

$$y^T b = y^T (A x_0) = a^T x_0 \leq \alpha \leq 0. \qquad \boxtimes$$