Corollary 7 Let $K \subseteq \mathbb{R}^n$ be a proper convex cone, let $A \in \mathbb{R}^{m \times n}$ be a matrix and $c \in \mathbb{R}^n$ be a vector. Then exactly one of the following two alternatives holds true:

(i) $\exists\, x \in K \setminus \{0\} : \; Ax = 0$ and $c^T x \geq 0$

(ii) $\exists\, y \in \mathbb{R}^m : \; A^T y - c \in \operatorname{int} K^*$.

§ 5 Duality theory

(P) $\quad p^* = \sup\limits_{\substack{x \in K \\ Ax = b}} c^T x \qquad\qquad$ (D) $\quad d^* = \inf\limits_{A^T y - c \in K^*} b^T y$.

Def. 1 The difference $d^* - p^*$ (which is nonnegative) is called duality gap.

Theorem 2 (a) (weak duality)

Suppose x is feasible for (P) and y is feasible for (D), then $c^T x \leq b^T y$. In particular, $p^* \leq d^*$.

(b) (complementary slackness) $p^* = d^*$

Let x be optimal for (P) and let y be optimal for (D).

Then $(A^T y - c)^T x = 0$.

(c) (optimality condition)

Let x be feasible for (P) and let y be feasible for (D).

(x, y) is optimal if and only if $(A^T y - c)^T x = 0$.

(d) (strong duality)

Suppose $d^* > -\infty$ and (D) is strictly feasible. Then there is an optimal solution of (P), i.e. there is a feasible x with $c^T x = p^*$, and $p^* = d^*$ holds.

Suppose $p^* < \infty$ and (P) is strictly feasible. Then there is a feasible solution y of (D) with $d^* = b^T y$, and $p^* = d^*$ holds.

<u>Remark</u> (c): is related to the <u>KKT-conditions</u>

(KKT = Karush-Kuhn-Tucker)

(d): if (P) & (D) are both strictly feasible, then we say that <u>Slater's condition</u> is fulfilled.

Proof

(a) - (c): $0 \leq (A^T y - c)^T x = y^T A x - c^T x = y^T b - c^T x.$

(d): Suppose $d^* > -\infty$ and (D) is strictly feasible.

To show: $\exists\, x^* \in K : \; Ax^* = b, \quad c^T x^* \geq d^*.$

1st case: $b = 0$.

Then $d^* = 0$. Set $x^* = 0$.

2nd case: $b \neq 0$.

Consider the non-empty convex set

$$M = \{ A^T y - c \;:\; y \in \mathbb{R}^m, \; b^T y \leq d^* \}.$$

We have

$$M \cap \operatorname{int} K^* = \emptyset.$$

[Otherwise there would exist a $y \in \cancel{\text{int}}\, K^*$ with $y^T b = d^*$ and an $\varepsilon > 0$ such that $B(y, \varepsilon) = \{ z : \|z - y\| \leq \varepsilon \} \subseteq K^*$.
For $y - \varepsilon \frac{b}{\|b\|} \in K^*$ we would get

$$b^T \left(y - \varepsilon \frac{b}{\|b\|} \right) = b^T y - \varepsilon \|b\| < d^*. \quad \lightning \;]$$

By the separation theorem, Theorem 4.6, there exists an $x \in \mathbb{R}^n \setminus \{0\}$ such that

$$\sup_{z \in M} x^T z \le \inf_{z \in \operatorname{int} K^*} x^T z = \inf_{z \in K^*} x^T z \qquad (*)$$

holds.

<u>Claim</u>: 1) $x \in K$

2) $\exists \mu > 0$: $Ax = \mu b$ and $c^T x \ge \mu d^*$

3) $x^* = \frac{x}{\mu}$ is optimal for (P)

3) follows from 1) and 2) and weak duality.

1) <u>Show</u>: $\inf_{z \in K^*} x^T z \ge 0$. Then $x \in (K^*)^* = K$.

Suppose $\exists\, z \in K^*$ with $x^T z < 0$. Then $x^T (tz) \to -\infty$ for $t \to +\infty$ in contradiction to $(*)$ and $d^* > -\infty$ $(M \neq \emptyset)$.

2) Since $0 \in K^*$, claim 1) implies $\inf_{z \in K^*} x^T z = 0$.

Hence, $\sup_{z \in M} x^T z \le 0$. That means

$b^T y \le d^*$ implies $x^T (A^T y - c) \le 0$ $(\Longleftrightarrow x^T A^T y \le x^T c)$.

So: The half space $\{y : b^T y \leq d^*\}$ is contained in the half space $\{y : (Ax)^T y \leq x^T c\}$.

Hence, b and Ax are linearly dependent and point in the same direction: $\exists \mu \geq 0 : Ax = \mu b$ and $\mu d^* \leq x^T c$ (**)

Assume $\mu = 0$. (D) is strictly feasible: $\exists y' : A^T y' - c \in \operatorname{int} K^*$

Using Exercise 2.4 gives

$$0 < (A^T y' - c)^T x = (y')^T A x - c^T x = (y')^T \mu b - c^T x \overset{(\mu=0)}{=} - c^T x,$$

and so $c^T x < 0$ in contradiction to (**). $\boxtimes$

The second statement of (d) follows from the first by taking the dual.

Some examples

Example 3 d^* not attained

$$p^* = \sup \langle E_{12}, X \rangle \qquad X \in S^2_{\geq 0} \qquad \langle E_{11}, X \rangle = 1, \ \langle E_{22}, X \rangle = 0$$

$$d^* = \inf y_1 \qquad \underbrace{y_1 E_{11} + y_2 E_{22} - E_{12}}_{= \begin{bmatrix} y_1 & -\frac{1}{2} \\ -\frac{1}{2} & y_2 \end{bmatrix}} \succeq 0$$

We have $p^* = d^* = 0$, but d^* is not attained.

Problem: (P) is not strictly feasible.

Example 4 positive duality gap.

$$p^* = \sup \langle -E_{11} - E_{22}, X\rangle \qquad\qquad d^* = \inf y_2$$

$$X \in S^3_{\succeq 0} \qquad\qquad y_1 E_{11} + y_2 (E_{22} + 2E_{13}) + E_{11} + E_{22} \succeq 0$$

$$\langle E_{11}, X\rangle = 0$$

$$\langle E_{22} + 2E_{13}, X\rangle = 1$$

$$= \begin{bmatrix} y_1+1 & 0 & y_2 \\ 0 & y_2+1 & 0 \\ y_2 & 0 & 0 \end{bmatrix} \succeq 0$$

Every feasible solution of (P) satisfies

$$X_{11} = 0 = X_{13}, \quad X_{22} = 1.$$

Hence, $p^* = -1$.

Every feasible solution of (D) satisfies $y_2 = 0$. Hence, $d^* = 0$.