Chapter IV First applications

§ 1 Robust optimization

Consider a practical LP in real world application

$$\max\{c^T x : Ax \geq b\}. \qquad (*)$$

Potential problems: data uncertainty

- some entries A_{ij}, b_j are only forecasted
  → prediction errors
- some entries A_{ij}, b_j come from measurements
  → measurement errors
- variables x_i cannot be implemented exactly
  → implementation errors.

Idea of robust optimization: Specify an uncertainty region Z and solve the robust counterpart of $(*)$

$$\max\{c^T x : A(z)x \geq b(z) \quad \forall z \in Z\}. \qquad (**)$$

If Z is nice, e.g. it can be described by an LMI, then the robust counterpart of the LP $(*)$ is an SDP.

Literature: Ben-Tal, El Ghaoui, Nemirovski – Robust Optimization, Princeton, 2009.

Here: only simple example, LP with one linear inequality:

(*) $\max\{c^T x : a^T x \geq b\}$, $c, a \in \mathbb{R}^n$, $b \in \mathbb{R}$.

uncertainty region $Z \subseteq \mathbb{R}^k$ unit ball in $\mathbb{R}^k$.

Robust counterpart of (*)

(**) $\max\{c^T x : a(z)^T x \geq b(z)\}$

with $a(z) = a_0 + \sum_{j=1}^{k} z_j a_j$, $b(z) = b_0 + \sum_{j=1}^{k} z_j b_j$,

where $z = (z_1, \ldots, z_k)^T \in Z$ and a_j, b_j are known beforehand.

Theorem 1 The robust counterpart (**) is equivalent to the following SDP:

$$\max\ c^T x$$
$$x \in \mathbb{R}^n,\quad Z \in S^{k+1}_{\geq 0},$$
$$2Z_{j\,k+1} = a_j^T x - b_j,\quad j \in [k]$$
$$\mathrm{Tr}(Z) \leq a_0^T x - b_0.$$

Proof Fix $x \in \mathbb{R}^n$.

Set $\alpha_j = a_j^T x - b_j$, $j = 0, \ldots, k$ and $\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix} \in \mathbb{R}^k$

Then constraint $\forall z \in Z$: $a(z)^T x \leq b(z)$ is equivalent to

$\alpha^T z \geq -\alpha_0 \quad \forall z \in Z$.

Now the question is whether

$$p^* = \min_{\|z\| \leq 1} \alpha^T z \geq -\alpha_0 \text{ ?}$$

Using Lemma III.3.4 $\left[\mathcal{L}^{n+1} = \{ (x,t) \in \mathbb{R}^{n+1} : \begin{bmatrix} tI_n & x \\ x^T & t \end{bmatrix} \succeq 0 \right]$

we see that

$$z \in Z \iff \|z\| \leq 1 \iff \begin{bmatrix} I_k & z \\ z^T & 1 \end{bmatrix} \succeq 0 .$$

So $$p^* = \min \left\langle \frac{1}{2}\begin{bmatrix} 0 & \alpha \\ \alpha^T & 0 \end{bmatrix}, X \right\rangle$$

$$X = \begin{bmatrix} I_k & z \\ z^T & 1 \end{bmatrix} \succeq 0,$$

$$z \in \mathbb{R}^k .$$

This SDP is strictly feasible and $p^* \geq -\infty$ is attained. Consider its dual: Strong duality yields:

$$p^* = d^* = \max \; -\mathrm{Tr}(Z)$$

$$Z \in S^{k+1}_{\succeq 0}$$

$$2 Z_{j\,k+1} = \alpha_j, \quad j = 1, \ldots, k .$$

Hence, the constraint

$$\forall z \in Z : \; a(z)^T x \leq b(z)$$

is equivalent to

$$\exists Z \in S^{k+1}_{\succeq 0} : \; 2 Z_{j,k+1} = \alpha_j, \; j \in [k] \text{ and } -\mathrm{Tr}(Z) \geq -\alpha_0 .$$

[Crucial: "$\forall$" is replaced by "$\exists$"].

Substitute x back into α_j:

$$(**) \iff \begin{array}{l} \max \ c^T x \\ x \in \mathbb{R}^n, \ Z \in S^{k+1}_{\geq 0} \\ 2 Z_{j\,k+1} = a_j^T x - b_j, \quad j = 1, \dots, k \\ \mathrm{Tr}(Z) \leq a_0^T x - b_0. \end{array}$$

$\boxtimes$

Remark: Matrix Z provides a proof that optimal vector x is a robust solution.

§2 Eigenvalue optimization

Theorem 1: The largest eigenvalue of a symmetric matrix $C \in S^n$ is given by

$$\lambda_{\max}(C) = \max_{\substack{X \in S^n_{\geq 0} \\ \langle I_n, X \rangle = 1}} \langle C, X \rangle = \min_{y I_n - C \succeq 0} y.$$

The smallest eigenvalue $\lambda_{\min}(C)$ of C we get by replacing max and min.