[Crucial: "$\forall$" is replaced by "$\exists$"].

Substitute x back into α_j:

$$(**) \iff \begin{array}{l} \max\ c^T x \\ x \in \mathbb{R}^n,\ Z \in S^{k+1}_{\geq 0} \\ 2 Z_{j\,k+1} = a_j^T x - b_j, \quad j = 1, \dots, k \\ \operatorname{Tr}(Z) \leq a_0^T x - b_0. \end{array}$$

$\boxtimes$

Remark: Matrix Z provides a proof that optimal vector x is a robust solution.

§2 Eigenvalue optimization

Theorem 1: The largest eigenvalue of a symmetric matrix $C \in S^n$ is given by

$$\lambda_{\max}(C) = \max_{\substack{X \in S^n_{\geq 0} \\ \langle I_n, X \rangle = 1}} \langle C, X \rangle = \min_{y I_n - C \succeq 0} y.$$

The smallest eigenvalue $\lambda_{\min}(C)$ of C we get by:

$$\lambda_{\min}(C) = \min_{\substack{X \in S^n_{\geq 0} \\ \langle I_n, X \rangle = 1}} \langle C, X \rangle = \max_{C - y I_n \succeq 0} y$$

<u>Proof</u> Consider the dual

$$d^* = \inf\{y : yI_n - C \succeq 0\}.$$

Then $y^* = \lambda_{max}(C)$ is a feasible solution because for $x \in \mathbb{R}^n \setminus \{0\}$ we have

$$x^T(y^* I_n - C)x = y^* x^T x - x^T C x \geq 0 \iff y^* \geq \frac{x^T C x}{x^T x}$$

and

$$y^* = \lambda_{max}(C) \overset{\text{Prop. II.3}}{=} \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^T C x}{x^T x}.$$ Hence, $d^* \leq \lambda_{max}(C)$

Consider the primal

$$p^* = \sup\{\langle C, X\rangle : X \succeq 0, \langle I, X\rangle\}.$$

Then an eigenvector x^* of $\lambda_{max}(C)$ with $\|x\| = 1$ defines a feasible solution $X^* = x^*(x^*)^T$ with objective value $\langle C, X^*\rangle = (x^*)^T C x^* = \lambda_{max}(C)$. Hence, $p^* \geq \lambda_{max}(C)$.

By weak duality

$$\lambda_{max}(C) \leq p^* \leq d^* \leq \lambda_{max}(C)$$

the theorem follows. ☒

Theorem 2 (Fan)

Let $C \in S^n$ be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Then,

$$\lambda_1 + \dots + \lambda_k = \max\{\langle C, X\rangle : X \in S^n, \ Tr(X) = k, \ I_n \succeq X \succeq 0\}$$

$$= \max\{\langle C, YY^T\rangle : Y \in \mathbb{R}^{n\times k}, \ Y^TY = I_k\}.$$

Recall: $\to$ OR (2015)

Theorem 3 (Minkowski; Krein-Milman)

Let $D \subseteq \mathbb{R}^n$ be a compact and convex set. Then

$D = \mathrm{conv}(\mathrm{ext}(D))$, where $\mathrm{ext}(D) = \{x \in D : x \text{ extreme point of } D\}$.

Lemma 4 Define

$$K_1 = \{X \in S^n : Tr(X) = k, \ I_n \succeq X \succeq 0\},$$

$$K_2 = \{YY^T : Y \in \mathbb{R}^{n\times k}, \ Y^TY = I_k\}.$$

Then $\mathrm{ext}(K_1) = K_2$. In particular, $\mathrm{conv}(K_2) = K_1$.

Proof For $P \in O(n)$ we have

$X \in K_1 \iff PXP^T \in K_1$ and $X \in \mathrm{ext}(K_1) \iff PXP^T \in \mathrm{ext}(K_1)$

Define the polytope

$$Q = \{ x \in [0,1]^n : e^T x = k \}, \quad \text{where} \quad e = (1, \dots, 1)^T.$$

Use spectral decomposition to write $X \in S^n$ as

$$X = PDP^T \quad \text{with } P \in O(n) \text{ and diagonal matrix } D.$$

Then,

$$X \in K_1 \iff D \in Q \quad \text{and} \quad X \in \operatorname{ext}(K_1) \iff D \in \operatorname{ext}(Q).$$

Clearly,

$$\operatorname{ext}(Q) = \{ x \in \{0,1\}^k : e^T x = k \}.$$

So, $X \in \operatorname{ext}(K_1) \iff X$ has exactly k eigenvalues equal to 1 and $n-k$ eigenvalues equal to 0.

That means X is a projection onto a k-dimensional subspace:
$\exists Y \in \mathbb{R}^{n \times k} : Y^T Y = I_k$ and $X = YY^T$ (check this!) $\boxtimes$

<u>Proof</u> (of Theorem 2)

Both maxima coincide: Follows immediately from OR:
Let $D \subseteq \mathbb{R}^n$, $D \neq \emptyset$, be convex and compact. We have for $c \in \mathbb{R}^n$:

$$\max \{ c^T x : x \in D \} = \max \{ c^T x : x \in \operatorname{ext}(D) \}.$$

Let p^* be the maximum.

$\underline{\lambda_1 + \dots + \lambda_k \leq p^*}$: Use spectral decomposition of C:

$$C = \sum_{i=1}^{n} \lambda_i u_i u_i^T \quad \text{and define } Y = [u_1 \dots u_k] \in \mathbb{R}^{n\times k}.$$

Then $YY^T \in K_2$ and $\langle C, YY^T\rangle = \lambda_1 + \dots + \lambda_k$.

$\underline{\lambda_1 + \dots + \lambda_k \geq p^*}$: Again use spectral decomposition of C:

$$C = PDP^T, \quad P \in O(n), \quad D \text{ diagonal matrix.}$$

We have for $YY^T \in K_2$:

$$\langle C, YY^T\rangle = \mathrm{Tr}(CYY^T) = \mathrm{Tr}(PDP^TYY^T)$$
$$= \mathrm{Tr}(DP^TYY^TP) = \langle D, \underbrace{P^TY(P^TY)^T}_{\in K_2}\rangle.$$

So,

$$p^* = \max\{\langle D, M\rangle = \sum_{i=1}^{n} \lambda_{ii} M_{ii} : M \in K_2\}.$$

For $M \in K_2$, the diagonal $\mathrm{diag}(M) = (M_{11}, \dots, M_{nn})$ lies in the polytope $Q = \{x \in [0,1]^n : e^Tx = k\}$ (check it!). Hence,

$$p^* \leq \max\{\sum_{i=1}^{n} \lambda_i x_i : x \in Q\} = \lambda_1 + \dots + \lambda_k. \qquad \boxtimes$$