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Convex Optimization

Winter Term 2015/16

— Exercise Sheet 7 —

Exercise 7.1. Let $G = (V, E)$ be a graph and $w \in \mathbb{R}^E_{\geq 0}$ a nonnegative weight function on the edges of G.

1. Show that $\mathrm{mc}(G, w) = \mathrm{sdp}(G, w)$ holds when G is a bipartite graph.
2. Determine the values of $\mathrm{mc}(\mathsf{K}_n, e)$ and $\mathrm{sdp}(\mathsf{K}_n, e)$ for the complete graph K_n on n vertices and the all-ones weight function $e = (1, \ldots, 1)^\mathsf{T}$

Exercise 7.2. Let $L_w \in \mathcal{S}^V$ be the Laplacian matrix of $G = (V, E)$ and $w \in \mathbb{R}^E$.

1. Show that: If $x \in \{-1, +1\}^V$, then
$$\frac{1}{4} x^\mathsf{T} L_w x = \frac{1}{2} \sum_{\{i,j\} \in E} w_{ij}(1 - x_i x_j).$$
2. Show that $L_w \succeq 0$ whenever $w \geq 0$.

Exercise 7.3.

1. Let $G = (V, E)$ be a *vertex-transitive* graph, i.e. for any two nodes $i, j \in V$ there exists an automorphism $\sigma \in \mathrm{Aut}(G)$ mapping i to j: $\sigma(i) = j$. Show:
$$\mathrm{sdp}(G, e) = \frac{|V|}{4} \lambda_{\max}(L_e).$$
2. Determine the ratio $\mathrm{mc}(C_5, e)/\mathrm{sdp}(C_5, e)$ for the 5-cycle graph C_5.

Exercise 7.4. Let $A \in \mathcal{S}^n_{\succeq 0}$ be a positive semidefinite matrix. Show the following identity:

$$\max\{\langle A, xx^\mathsf{T} \rangle : x \in \{-1, +1\}^n\} = \max\left\{\frac{2}{\pi}\langle A, \arcsin X \rangle : X \in \mathcal{S}^n_{\succeq 0},\ X_{ii} = 1, i = 1, \ldots, n\right\}.$$

Hand-in: Until Tuesday, 7th December, 2pm at the "Convex optimization" mailbox in room 3.01 (Studierendenarbeitsraum) of the Mathematical Institute. Please add your name, student number, and group number to your solution sheet.