Consider SDP relaxation of the MAX CUT QP. Since it does not involve linear terms we can use an SDP relaxation of size $|V|$ (instead of size $|V|+1$):

$$\mathrm{sdp}(G,w) = \max \frac{1}{2} \sum_{\{i,j\}\in E} w_{ij}(1 - X_{ij})$$
$$X \in \mathcal{S}^V_{\succeq 0}$$
$$X_{ii} = 1 \quad \text{for } i \in V.$$

Clear: $\mathrm{mc}(G,w) \leq \mathrm{sdp}(G,w)$.

Every $x \in \{-1,+1\}^V$ is a feasible solution for the MAX CUT QP. Then $X = xx^T$ is a feasible solution for the SDP relaxation.

If $\operatorname{rank} X = 1$ and X is feasible for SDP-relaxation, then $X = xx^T$ where $x \in \{-1,+1\}^V$. ($\rightarrow$ Exercise 3.2)

How good is the SDP relaxation?

Theorem 5 (Goemans, Williamson, 1995)

$$\mathrm{sdp}(G,w) \geq \mathrm{mc}(G,w) \geq 0{,}878\ldots\ \mathrm{sdp}(G,w).$$

Proof The proof of the inequality is algorithmic. It will yield a randomised polynomial time algorithm to approximate MAXCUT within a factor of 0,878...

Algorithm 1. Solve sdp(G, w). Let X be an optimal solution.

2. Perform a Cholesky decomposition of X. This gives normal vectors $v_i \in \mathbb{R}^V$, $i \in V$, with
$$X_{ij} = v_i^T v_j, \quad i,j \in V.$$

randomized rounding

3. Choose $r \in \mathbb{R}^V$, $\|r\| = 1$, randomly from the probability distribution on the unit sphere which is invariant under orthogonal transformation.

4. Define a cut $\delta(S)$ by
$$i \in S \Leftrightarrow \operatorname{sign}(v_i^T r) \geq 0 \quad \text{for } i \in V.$$
That means $x_i = \begin{cases} \operatorname{sign}(v_i^T r), & \operatorname{sign}(v_i^T r) \neq 0 \\ 1, & \text{otherwise.} \end{cases}$

5. If $w(\delta(S)) < 0,878\ldots$ sdp(G, w): go back to step 3.

Now show: $\mathbb{E}[w(\delta(S))] \geq 0,878\ldots \operatorname{sdp}(G, w)$.

Lemma 6 (Grothendieck's identity)

Let $u, v \in \mathbb{R}^d$ be unit vectors. Let $r \in \mathbb{R}^d$ be a random unit vector chosen from the $O(d)$-invariant probability distribution on the unit sphere. Then

(i) $\mathbb{P}\left[\operatorname{sign}(u^T r) \neq \operatorname{sign}(v^T r)\right] = \dfrac{\arccos(u^T v)}{\pi}$

(ii) $\mathbb{E}\left[\operatorname{sign}(u^T r)\operatorname{sign}(v^T r)\right] = \dfrac{2}{\pi}\arcsin(u^T v)$.

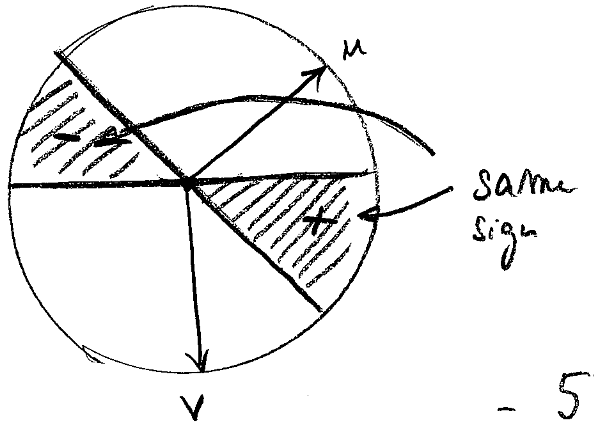
Proof (i) If $u = v$, then $\arccos(u^T v) = 0$

If $u = -v$, then $\arccos(u^T v) = \pi$.

In both cases, the equality in (i) is obvious.

Now assume that $W = \operatorname{span}\{u, v\}$ has dimension 2.
Project r orthogonally onto W: this gives an $s \in W$ such that $u^T r = u^T s$, $v^T r = v^T s$. The unit vector $\frac{s}{\|s\|}$ is uniformly distributed on the unit circle by the $O(d)$-invariance of the probability distribution. Hence,

$$\mathbb{P}\left[\operatorname{sign}(u^T r) \neq \operatorname{sign}(v^T r)\right] = 2 \cdot \frac{1}{2\pi}\arccos(u^T v)$$



(ii) $\mathbb{E}[\operatorname{sign}(u^T r)\operatorname{sign}(v^T r)]$

$$= (+1)\,\mathbb{P}[\operatorname{sign}(u^T r) = \operatorname{sign}(v^T r)] + (-1)\,\mathbb{P}[\operatorname{sign}(u^T r) \neq \operatorname{sign}(v^T r)]$$

$$= 1 - 2\,\mathbb{P}[\operatorname{sign}(u^T r) \neq \operatorname{sign}(v^T r)]$$

$$= 1 - 2\,\frac{\arccos(u^T v)}{\pi}$$

$$= \frac{2}{\pi}\arcsin(u^T v) \quad \text{because } \arcsin t + \arccos t = \frac{\pi}{2}.$$

$\boxtimes$

Back to the proof of Theorem 5:

Let X be an optimal solution of $sdp(G,w)$ and let $v_i \in \mathbb{R}^V$, $i \in V$, be unit vectors with $X_{ij} = v_i^T v_j$ constructed in Step 2.

Consider the cut $\delta(S)$ determined in Step 4. Then:

$$\mathbb{E}[w(\delta(S))] = \sum_{\{i,j\}\in E} w_{ij}\,\mathbb{P}[\{i,j\} \in \delta(S)]$$

$$= \sum_{\{i,j\}\in E} w_{ij}\,\mathbb{P}[\operatorname{sign}(v_i^T r) \neq \operatorname{sign}(v_j^T r)]$$

$$= \sum_{\{i,j\}\in E} w_{ij}\,\frac{\arccos(v_i^T v_j)}{\pi}$$

$$= \sum_{\{i,j\}\in E} w_{ij}\left(\frac{1 - v_i^T v_j}{2}\right)\cdot\underbrace{\left(\frac{2}{\pi}\,\frac{\arccos(v_i^T v_j)}{1 - v_i^T v_j}\right)}_{\geq 0{,}878\ldots}$$

$$\geq 0{,}878\ldots\ sdp(G,w)$$

because $\min_{t\in[-1,1]} \frac{2}{\pi}\frac{\arccos t}{1-t} = 0{,}878\ldots$

Since $mc(G,w) \geq \mathbb{E}[w(\delta(S))]$ Theorem 5 is proven. $\boxtimes$