Remarks:

- probability distribution on sphere S^{n-1} invariant under $O(n)$:

  choose $X_1, \dots, X_n \sim N(0,1)$ from normal distribution

  (with density $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$)

  then $r = \frac{1}{\|x\|} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in S^{n-1}$ is chosen randomly

  according to the $O(n)$-invariant probability distribution.

- one can derandomize the algorithm.

- approximation factor might be optimal

  ($\rightarrow$ Khot's unique games conjecture $\rightarrow$ seminar)

Some words about the running time of the Goemans Williamson algorithm:

- algorithm is randomized; it's a Las Vegas algorithm (when it terminates, then the output is correct, running time is a random variable)

Corollary 6 Let $\varepsilon > 0$. Let $\delta(S)$ be the cut defined in Step 4. Then

$$\mathbb{P}\left[w(\delta(S)) < (1-\varepsilon)\, \mathbb{E}[w(\delta(S))] \right] \leq 1 - \frac{0{,}4\,\varepsilon}{0{,}6+\varepsilon}$$

Proof Define $W = \sum_{\{i,j\} \in E} w_{ij}$ and $\alpha = \mathbb{E}[w(\delta(S))] / W$.

We want to bound the probability

$$p = \mathbb{P}[w(\delta(S)) < (1-\varepsilon)\alpha W].$$

Clearly,

$$\mathbb{E}[w(\delta(S))] \leq p \cdot (1-\varepsilon)\alpha W + (1-p) W,$$

hence $p \leq \dfrac{1-\alpha}{1-\alpha+\alpha\varepsilon} = 1 - \dfrac{\alpha\varepsilon}{1-\alpha+\alpha\varepsilon}$

Now we need an upper and a lower bound for α. Upper bound: $\alpha \leq 1$. Lower bound:

$$\mathbb{E}[w(\delta(S))] = \alpha W \geq 0{,}878\ldots \, sdp(G,w) \geq 0{,}878\ldots \, mc(G,w)$$

$$\geq \frac{0{,}878\ldots W}{2} \quad \text{because} \quad mc(G,w) \geq \frac{W}{2} \quad (\text{why?})$$

So: $\alpha \geq \dfrac{0{,}878\ldots}{2} \geq 0{,}4$

Therefore: $p \leq 1 - \dfrac{0{,}4\varepsilon}{1-0{,}4+\varepsilon}$. $\boxtimes$

Define $C = \dfrac{0{,}4\varepsilon}{0{,}6+\varepsilon}$. If we loop through steps 1-5 at least $\frac{1}{C}$ times, then we found a cut of quality $(1-\varepsilon)\, 0{,}878\ldots \, mc(G,w)$ with probability $\geq 1-(1-C)^{1/C} \geq 1 - \frac{1}{e} = 0{,}632\ldots$