§2 Eigenvalue interpretation of the SDP relaxation for MAX CUT

<u>Def. 1</u> Let $G = (V, E)$ be a graph with edge weights $w \in \mathbb{R}^E$. Its <u>Laplacian matrix</u> $L_w \in \mathbb{R}^{V \times V}$ is defined as follows:

$$(L_w)_{ii} = \sum_{\substack{j \in V \\ \{i,j\} \in E}} w_{ij} \quad , \quad i \in V$$

$$(L_w)_{ij} = \begin{cases} -w_{ij} & , \text{ if } \{i,j\} \in E \\ 0 & , \text{ otherwise} \end{cases} \quad , \quad i \neq j.$$

<u>Lemma 2</u> (a) For all $x \in \{-1, +1\}^V$ we have

$$\frac{1}{4} x^T L_w x = \frac{1}{2} \sum_{\{i,j\} \in E} w_{ij} (1 - x_i x_j) \qquad (*)$$

(b) If $w \in \mathbb{R}^E_{\geq 0}$, then $L_w \in S^V_{\succeq 0}$.

<u>Proof</u> $\rightarrow$ exercise.

$(*)$ gives: $\quad mc(G, w) = \max \{ \frac{1}{4} x^T L_w x : x_i^2 = 1, i \in V \}$

$\qquad sdp(G, w) = \max \{ \frac{1}{4} \langle L_w, X \rangle : X \succeq 0, X_{ii} = 1, i \in V \}.$

Its dual semidefinite program is

$$\mathrm{sdp}(G,w) = \min \left\{ \sum_{i\in V} y_i : \mathrm{Diag}(y) - \frac{1}{4} L_w \succeq 0 \right\},$$

where $\mathrm{Diag}(y) \in S^V$ is the diagonal matrix with entries y_i, $i\in V$, on the diagonal.

Note that strong duality holds: both programs are strictly feasible: choose $X = I$ or $y = Ce$, with C big enough.

<u>Theorem 3</u> $$\mathrm{sdp}(G,w) = \min_{\substack{u\in\mathbb{R}^V \\ \sum_{i\in V} u_i = 0}} \frac{|V|}{4} \lambda_{max}(L_w + \mathrm{Diag}(u)) \qquad (**)$$

<u>Proof</u> Use Theorem IV.2.1 to observe that

$$\lambda_{max}(L_w + \mathrm{Diag}(u)) = \min_{\substack{t\in\mathbb{R} \\ tI - (L_w + \mathrm{Diag}(u)) \succeq 0}} t$$

holds. So the RHS of $(**)$ equals

$$\min_{\substack{t\in\mathbb{R},\ u\in\mathbb{R}^n,\ \sum_{i\in V} u_i = 0 \\ tI - (L_w + \mathrm{Diag}(u)) \succeq 0.}} \frac{|V|}{4} t \qquad (***)$$

Now consider a feasible solution $y \in \mathbb{R}^V$ of the dual formulation of $\mathrm{sdp}(G,w)$.

Define a feasible solution for $(***)$:

$$t = \frac{4}{|V|} \sum_{i \in V} y_i \qquad \text{and} \qquad u = te - 4y \qquad \text{with } e = (1, \dots, 1)^T$$

It is indeed feasible:

$$\frac{|V|}{4} t = \sum_{i \in V} y_i ,$$

$$\sum_{i \in V} u_i = e^T u = t e^T e - 4 e^T y = t |V| - 4 e^T y = 0,$$

$$tI - (L_w + \mathrm{Diag}(u)) = 4\,\mathrm{Diag}(y) - L_w \succeq 0.$$

Conversely, let (t, u) be a feasible solution for $(***)$.
Define $y = \frac{1}{4}(te - u)$ and check (easy) that this is feasible for the dual formulation $\boxtimes$

__Corollary 4__ $mc(G, w) \leq sdp(G, w) \leq \frac{|V|}{4} \lambda_{max}(L_w)$.

__Proof__ Set $u = 0$ in Theorem 3.

§ 3 Little Grothendieck inequality / Approximation algorithm of Nesterov

general QP with $-1/+1$ constraint:

$$qp(A) = \max \left\{ \sum_{i,j=1}^{n} A_{ij} x_i x_j : x_i^2 = 1, \ i = 1, \dots, n \right\}$$

for a symmetric matrix $A \in S^n$.

If $A = \frac{1}{4} L_w$, then $qp(A) = mc(G, w)$.

SDP relaxation:

$$sdp(A) = \max \{ \langle A, X \rangle : X \succeq 0, \ X_{ii} = 1, \ i = 1, \dots, n \}.$$

Goemans-Williamson: $sdp(A) \geq qp(A) \geq 0{,}878\ldots \, sdp(A)$
if $A = \frac{1}{4} L_w$ and $w \geq 0$.

Now: Consider the case $A \succeq 0$ ($\frac{1}{4} L_w \succeq 0$ if $w \geq 0$ by Lemma 2.2 (b))

<u>Theorem 1</u> (Nesterov 1997; "little Grothendieck inequality")

If $A \in S^n_{\succeq 0}$, then

$$sdp(A) \geq qp(A) \geq \frac{2}{\pi} sdp(A), \quad \text{where } \frac{2}{\pi} = 0{,}636\ldots$$

<u>Proof</u> Exactly the same algorithm (Step 1 – 4) as the Goemans-Williamson algorithm but different analysis:

Consider an optimal solution $X \in S^n_{\succeq 0}$ of the SDP relaxation and unit vectors $v_1, \dots, v_n$ with $X_{ij} = v_i^T v_j$.

Then $x_i = \operatorname{sign}(v_i^T r)$ where r was a random unit vector.

We have

$$qp(A) \geq \mathbb{E}\left[\sum_{i,j=1}^n A_{ij} x_i x_j\right] \overset{\substack{\text{linearity} \\ \text{of expectation}}}{=} \sum_{i,j=1}^n A_{ij}\, \mathbb{E}[x_i x_j]$$

$$\overset{\substack{\text{Grothendieck} \\ \text{identity}}}{=} \frac{2}{\pi} \sum_{i,j=1}^n A_{ij} \arcsin X_{ij} = \frac{2}{\pi} \langle A, \arcsin X \rangle,$$

where $\arcsin X \in S^n$ where $(\arcsin X)_{ij} = \arcsin X_{ij}$.

Recall: series expansion of $\arcsin t = t + \frac{1}{2}\frac{t^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{t^5}{5} + \dots$

all coefficients are non negative

Recall: Schur-Hadamard product $(X \circ Y)_{ij} = X_{ij} Y_{ij}$.

and $X \succeq 0, Y \succeq 0 \Rightarrow X \circ Y \succeq 0$. (Chapter III. 2)

Hence: If $X \succeq 0$, so is $\arcsin X - X \succeq 0$

Since $A \succeq 0$ and the PSD cone is self-dual:

$$\frac{2}{\pi} \langle A, \arcsin X \rangle \geq \frac{2}{\pi} \langle A, X \rangle = \frac{2}{\pi}\, sdp(A). \qquad \boxtimes$$