§ 4 Grothendieck's inequality

<u>Def. 1</u> For $A \in \mathbb{R}^{m \times n}$ define the quadratic program

$$\|A\|_{\infty \to 1} = \max \Big\{ \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_i y_j : \begin{array}{l} x_i^2 = 1, \ i = 1, \dots, m, \\ y_j^2 = 1, \ j = 1, \dots, n \end{array} \Big\}.$$

[$\|A\|_{\infty \to 1}$ is the norm of A interpreted as a linear operator from ℓ_∞^n to ℓ_1^m.]

Computing $\|A\|_{\infty \to 1}$ is NP-hard since for G, w one can find matrix A (in polynomial time) so that $\|A\|_{\infty \to 1} = mc(G, w)$. Also $qp(A)$, $A \succeq 0$, from § 3 is a special case.

<u>Def. 2</u> SDP relaxation of $\|A\|_{\infty \to 1}$:

$$sdp_{\infty \to 1}(A) = \max \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} u_i^T v_j$$
$$u_i \in \mathbb{R}^{m+n}, \ v_j \in \mathbb{R}^{m+n}$$
$$\|u_i\| = 1, \quad i \in [m]$$
$$\|v_j\| = 1, \quad j \in [n].$$

<u>Theorem 3</u> (Grothendieck's inequality)

$\exists$ constant K $\forall A \in \mathbb{R}^{m \times n}$:

$$\|A\|_{\infty \to 1} \le sdp_{\infty \to 1}(A) \le K \, \|A\|_{\infty \to 1}.$$

Def. 4 The smallest constant K so that Grothendieck's inequality holds, is called the Grothendieck constant K_G.

At the moment: Exact value of K_G is not known.

We know $K_G \in (1{,}676\ldots, 1{,}782\ldots)$

We will prove that $K_G \leq \frac{\pi}{2\ln(1+\sqrt{2})} = 1{,}782\ldots$.

(Will give approx. algo. for $\|A\|_{\infty\to 1}$ with approx. factor $1/1{,}782\ldots = 0{,}561\ldots$)

Proof (Theorem 3 by Krivine, 1979)

Again: approximation algorithm with randomized rounding

1. Solve $sdp_{\infty\to 1}(A)$. Let $u_1,\ldots,u_m, v_1,\ldots,v_n \in S^{m+n-1}$ be the optimal unit vectors
2. Apply Krivine's trick (Lemma 5) and use vectors u_i, v_j to create new unit vectors $u_1',\ldots,u_m', v_1',\ldots,v_n'$.
3. Choose $r \in S^{m+n-1}$ randomly
4. Round: $x_i = \operatorname{sign}((u_i')^T r)$, $i \in [m]$
   $y_j = \operatorname{sign}((v_j')^T r)$, $j \in [n]$

expected quality of the outcome:

$$\|A\|_{\infty\to1} \geq \mathbb{E}\Big[\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}\, x_i x_j\Big]$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}\, \mathbb{E}\big[\operatorname{sign}((u_i')^T r)\operatorname{sign}((v_j')^T r)\big]$$

$$\overset{\text{Lemma 5}}{=} \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}\, \beta\, u_i^T v_j = \beta\, \mathrm{sdp}_{\infty\to1}(A),$$

where $\beta = \frac{2\ln(1+\sqrt{2})}{\pi}$, thus $K_G \leq \beta^{-1}$. $\boxtimes$

<u>Lemma 5</u> (Krivine's trick)

Let $u_1,\dots,u_m, v_1,\dots,v_n \in S^{m+n-1}$ be given. Then there are $u_1',\dots,u_m', v_1',\dots,v_n' \in S^{m+n-1}$ so that

$$\mathbb{E}\big[\operatorname{sign}((u_i')^T r)\operatorname{sign}((v_j')^T r)\big] = \beta\, u_i^T v_j,$$

with $\beta = \frac{2}{\pi}\ln(1+\sqrt{2}) = 0{,}561\ldots$.

For the proof of Lemma 5 we need to use <u>tensor products</u>:

$\mathbb{R}^n$ is n-dim. Euclidean space with inner product x^Ty and orthonormal basis $e_1,\dots,e_n$. Define the k-th tensor product of $\mathbb{R}^n$ as follows: It is denoted by $(\mathbb{R}^n)^{\otimes k}$ and it is a Euclidean vector space of dimension n^k with orthonormal basis $e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_k}$, $i_j\in\{1,\dots,n\}$.

In particular:

$$(e_{i_1} \otimes \cdots \otimes e_{i_k})^T (e_{j_1} \otimes \cdots \otimes e_{j_k}) = \prod_{r=1}^{k} e_{i_r}^T e_{j_r}$$

and for $v \in \mathbb{R}^n$ with $v = v_1 e_1 + \cdots + v_n e_n$ we define

$$v^{\otimes k} \in (\mathbb{R}^n)^{\otimes k} \quad \text{by} \quad v^{\otimes k} = (v_1 e_1 + \cdots + v_n e_n) \otimes \cdots \otimes (v_1 e_1 + \cdots + v_n e_n)$$

$$\xrightarrow[\text{law}]{\text{distributive}} = \sum_{i_1, \ldots, i_k} v_{i_1} \cdots v_{i_k} \, e_{i_1} \otimes \cdots \otimes e_{i_k}$$

For $v, w \in \mathbb{R}^n$: $(v^{\otimes k})^T w^{\otimes k} = (v^T w)^k$.

<u>Proof</u> (of Lemma 5)

Define the function $E : [-1, +1] \to [-1, +1]$ by $E(t) = \frac{2}{\pi} \arcsin t$.

Due to Grothendieck's identity (Lemma 1.6):

$$E((u_i')^T v_j') = \mathbb{E}[\operatorname{sign}((u_i')^T r) \operatorname{sign}((v_j')^T r)]$$

$$\overset{!}{=} \beta \, u_i^T v_j .$$

<u>Idea</u>: To find β, u_i', v_j' invert function E:

$$(u_i')^T v_j' = E^{-1}(\beta u_i^T v_j)$$

and use series expansion $E^{-1}(t) = \sum_{r=0}^{\infty} g_{2r+1} t^{2r+1}$ which is valid for all $t \in [-1, 1]$ (note: E is odd, $E(-t) = -E(t)$, so E^{-1} also odd).

Define the infinite-dimensional Hilbert space

$$H = \bigoplus_{r=0}^{\infty} (\mathbb{R}^{m+n})^{\otimes 2r+1}$$

Define $u_i', v_j' \in H$ componentwise

$$(u_i')_r = \mathrm{sign}(g_{2r+1}) \sqrt{|g_{2r+1}|\beta^{2r+1}}\; u^{\otimes 2r+1}$$

$$(v_j')_r = \sqrt{|g_{2r+1}|\beta^{2r+1}}\; v^{\otimes 2r+1}$$

Then:

$$(u_i')^T v_j' = \sum_{r=0}^{\infty} g_{2r+1}\beta^{2r+1}(u^T v)^{2r+1} = E^{-1}(\beta u_i^T v_j).$$

β is defined by the condition

$$1 = (u_i')^T u_i = (v_j')^T v_j = \sum_{r=0}^{\infty} |g_{2r+1}|\beta^{2r+1}.$$

We have $E(t) = \frac{2}{\pi}\arcsin t$

$$E^{-1}(t) = \sin\left(\frac{\pi}{2}\right) = \sum_{r=0}^{\infty} \underbrace{\frac{(-1)^{2r+1}}{(2r+1)!}\left(\frac{\pi}{2}\right)^{2r+1}}_{g_{2r+1}} t^{2r+1}.$$

So:

$$1 = \sum_{r=0}^{\infty} \left|\frac{(-1)^{2r+1}}{(2r+1)!}\left(\frac{\pi}{2}\right)^{2r+1}\right|\beta^{2r+1} = \sinh\left(\frac{\pi}{2}\beta\right),$$

and

$$\beta = \frac{2}{\pi}\operatorname{arsinh} 1 = \frac{2}{\pi}\ln(1+\sqrt{2}) \quad \text{since } \operatorname{arsinh} t = \ln(t+\sqrt{t^2+1}).$$

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