From this representation it follows that F is convex since it is a maximum of a set of convex functions

$$g_A : S^n \to \mathbb{R}, \quad g_A(X) = f(\operatorname{diag}(AXA^T)), \quad A \in O(n).$$

"$\le$" in $(*)$: Spectral decomposition of $X = A^T \operatorname{Diag}(\lambda_1, \dots, \lambda_n) A$, with $A \in O(n)$. Then

$$F(X) = f(\lambda_1, \dots, \lambda_n) = f(\operatorname{diag}(AXA^T)).$$

"$\ge$" in $(*)$:
$$\max_{A \in O(n)} f(\operatorname{diag}(AXA^T)) \le \max_{Y \in SH(X)} f(\operatorname{diag}(Y))$$
$$\overset{\text{Thm. 4}}{=} \max_{x \in \Pi(\lambda_1, \dots, \lambda_n)} f(x)$$
$$\overset{\substack{f \text{ convex} \\ \text{and symmetric}}}{=} f(\lambda_1, \dots, \lambda_n)$$
$$= F(X). \qquad \boxtimes$$

Maximum of a convex function over convex polytope is attained at a vertex.

Corollary 5 $F(X) = \begin{cases} -(\det X)^{1/n} & \text{if } X \succeq 0 \\ 0 & \text{otherwise} \end{cases}$

is convex and spectral. (Minkowski's determinant inequality).

<u>Proof</u> to show: $f(x_1, \dots, x_n) = -\left(\prod_{i=1}^{n} x_i\right)^{1/n}$

is symmetric and convex.

symmetry: obvious

convexity: follows from the inequality between the arithmetic mean and the geometric mean (AM-GM inequality).

<u>Lemma 6</u> AM-GM inequality

(a) For $x_1, \dots, x_n \geq 0$:

$$\left(\prod_{i=1}^{n} x_i\right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i$$

with equality iff $x_1 = \dots = x_n$

(geometric interpretation: Volume of parallelepiped with side length $S = \sum_{i=1}^{n} x_i$ is maximal iff $x_1 = \dots = x_n = \frac{S}{n}$.)

(b) For $x_1, \dots, x_n > 0$, $y_1, \dots, y_n > 0$:

$$\left(\prod_{i=1}^{n} x_i\right)^{1/n} + \left(\prod_{i=1}^{n} y_i\right)^{1/n} \leq \left(\prod_{i=1}^{n} (x_i + y_i)\right)^{1/n}$$

with equality iff the vectors $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are linearly dependent.

Proof (a) George Pólya's (1887 - 1985) best mathematical dream: We have $1+y \le e^y$ for all $y \in \mathbb{R}$.

So $y \le e^{y-1}$ and $y^{1/n} \le \left(e^{y-1}\right)^{1/n} = e^{y/n - 1/n}$.

Define $y_i = \dfrac{x_i}{\frac{1}{n}\sum_{j=1}^{n} x_j}$. Then

$$\left(\prod_{i=1}^{n} y_i\right)^{1/n} \le \left(\prod_{i=1}^{n} e^{y_i - 1}\right)^{1/n} = e^{\frac{1}{n}\sum_{i=1}^{n} y_i - 1} = e^0 = 1.$$

So the inequality follows. Note that $1+y = e^y$ iff $y=0$.

(b)
$$\frac{\left(\prod_{i=1}^{n} x_i\right)^{1/n} + \left(\prod_{i=1}^{n} y_i\right)^{1/n}}{\left(\prod_{i=1}^{n}(x_i+y_i)\right)^{1/n}} = \left(\prod_{i=1}^{n} \frac{x_i}{x_i+y_i}\right)^{1/n} + \left(\prod_{i=1}^{n} \frac{y_i}{x_i+y_i}\right)^{1/n}$$

$$\overset{(a)}{\le} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_i+y_i} + \frac{1}{n}\sum_{i=1}^{n} \frac{y_i}{x_i+y_i}$$

$$= 1.$$

Equality holds iff (by (a)):

$$\frac{x_1}{x_1+y_1} = \frac{x_2}{x_2+y_2} = \dots = \frac{x_n}{x_n+y_n} \quad \text{and} \quad \frac{y_1}{x_1+y_1} = \dots = \frac{y_n}{x_n+y_n}.$$

Hence, $\dfrac{x_i}{x_j} = \dfrac{x_i+y_i}{x_j+y_j} = \dfrac{y_i}{y_j}$ for all $i \neq j$. So $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ are lin. dep. $\boxtimes$

end of proof of Corollary 5:

We have for $x, y \in \mathbb{R}^n$, $\alpha \in [0,1]$ that $f(\alpha x + (1-\alpha) y)$ equals

$$-\left(\prod_{i=1}^{n} (\alpha x_i + (1-\alpha) y_i)\right)^{1/n} \overset{L.6(b)}{\leq} -\left(\prod_{i=1} \alpha x_i\right)^{1/n} - \left(\prod (1-\alpha) y_i\right)^{1/n}$$

$$= \alpha f(x) + (1-\alpha) f(y).$$

$\boxtimes$

§2 MAXDET optimization

Theorem 1 The set

$$\mathcal{D}^n = \{ (X, s) \in \mathcal{S}^n \times \mathbb{R} : X \succeq 0, s \geq 0, (\det X)^{1/n} \geq s \}$$

is a proper convex cone.

Proof • $\mathcal{D}^n$ is a convex cone.

Let $\alpha \geq 0$, $(X,s), (Y,t) \in \mathcal{D}^n$ be given:

Then $(\alpha X, \alpha s) \in \mathcal{D}^n$ because $\alpha X \succeq 0$, $\alpha s \geq 0$ and $(\det \alpha X)^{1/n} = \alpha (\det X)^{1/n} \geq \alpha s$.

Then $(X+Y, s+t) \in \mathcal{D}^n$ because $X+Y \succeq 0$, $s+t \geq 0$ and $(\det(X+Y))^{1/n} \geq (\det X)^{1/n} + (\det Y)^{1/n} \geq s+t$.

Corollary 1.5.

- $\mathcal{D}^n$ is pointed:

  Suppose $(X,s), (-X,-s) \in \mathcal{D}^n$. We have $X=0$ since $S^n_{\geq 0}$ is pointed. We have $s=0$ since $\mathbb{R}_{\geq 0}$ is pointed.

- $\operatorname{int} \mathcal{D}^n \neq \emptyset$:

  Consider an open neighborhood of $(I_n, 1)$.

- $\mathcal{D}^n$ is closed:

  Clear because det is continuous.

<u>Theorem 2</u> The dual cone of $\mathcal{D}^n$ is

$$(\mathcal{D}^n)^* = \{ (Y,t) \in S^n \times \mathbb{R} : \ Y \succeq 0 , \ (\det Y)^{1/n} \geq -\tfrac{t}{n} \}.$$

<u>Lemma 3</u> Let $X \in S^n_{\geq 0}$ be a positive semidefinite matrix.

Then
$$\operatorname{Tr}(X) - n (\det X)^{1/n} \geq 0, \qquad (*)$$

where equality holds iff X is a multiple of the identity matrix.

<u>Proof</u> We may assume that X is a diagonal matrix $X = \operatorname{Diag}(x_1, \dots, x_n)$. Then

$$(*) \iff \sum_{i=1}^n x_i - n \left(\prod_{i=1}^n x_i \right)^{1/n} \geq 0 \quad \text{(AM-GM inequality)}.$$

$\boxtimes$