Proof (Thm. 2)

"$\supseteq$": Let $(Y,t) \in S^n \times \mathbb{R}$ with $Y \succeq 0$ and $(\det Y)^{1/n} \geq -\frac{t}{n}$, and let $(X,s) \in \mathcal{D}^n$. Then

$$\langle (X,s), (Y,t) \rangle = \operatorname{Tr}(XY) + st.$$

Would like to apply Lemma 3 but XY is generally not even symmetric. Solution: Use Cholesky factorization: $X = LL^T$. Then

$$\operatorname{Tr}(XY) = \operatorname{Tr}(LL^TY) = \operatorname{Tr}(L^TYL) \text{ and } L^TYL \succeq 0.$$

So

$$\begin{aligned}
\operatorname{Tr}(XY) + st &= \operatorname{Tr}(L^TYL) + st \\
&\overset{L.3}{\geq} n \det(L^TYL)^{1/n} + st \\
&= n \det(LL^T)^{1/n} (\det Y)^{1/n} + st \\
&= n (\det X)^{1/n} (\det Y)^{1/n} + st \\
&\geq n \cdot s \cdot \left(-\frac{t}{n}\right) + st \\
&\geq 0.
\end{aligned}$$

"$\subseteq$": Consider $(Y,t) \in (\mathcal{D}^n)^*$. For $(X,0) \in \mathcal{D}^n$ we have

$$0 \leq \langle (X,0), (Y,t) \rangle = \langle X, Y \rangle + 0 \cdot t = \langle X, Y \rangle.$$

Since $(S^n_{\succeq 0})^* = S^n_{\succeq 0}$, it follows that $Y \succeq 0$.

Now consider $(X, (\det X)^{1/n}) \in \mathcal{D}^n$ with $X \succ 0$. Then

$$0 \leq \mathrm{Tr}(XY) + (\det X)^{1/n}\, t.$$

Therefore,

$$-t \leq \frac{\mathrm{Tr}(XY)}{(\det X)^{1/n}}.$$

Minimise the function, depending on X, of the RHS.

<u>1st case</u>: Y not positive definite, only positive semidefinite

Then the infimum is zero: Let $u_1 \in \mathbb{R}^n$ be a unit vector s.t. $u_1^T Y u_1 = 0$. Complete u_1 to an ONB $u_1, \dots, u_n$ and define $X = u_1 u_1^T + \varepsilon \sum_{i=2}^n u_i u_i^T$ for $\varepsilon > 0$. Then,

$$0 \leq \frac{\mathrm{Tr}(XY)}{(\det X)^{1/n}} = \frac{u_1^T Y u_1 + \varepsilon \sum_{i=2}^n u_i^T Y u_i}{(\varepsilon^{n-1})^{1/n}}$$

$$\leq \frac{0 + \varepsilon(n-1)\lambda_{max}(Y)}{\varepsilon^{(n-1)/n}} = \varepsilon^{\frac{1}{n}}(n-1)\lambda_{max}(Y) \xrightarrow{\varepsilon \to 0} 0.$$

<u>2nd case</u>: Y is positive definite

Then the minimum is attained at $X = Y^{-1}$ and it is equal to $(\det Y)^{1/n} \cdot n$. This follows from the AM-GM inequality (Lemma 3) as follows:

By Lemma 3 (see also proof of "$\supseteq$"):

$$\mathrm{Tr}(XY) \geq n(\det X)^{1/n}(\det Y)^{1/n}$$

with equality iff XY is a multiple of the identity.

Hence,

$$-t \leq n(\det Y)^{1/n} \Rightarrow (\det Y)^{1/n} \geq -\frac{t}{n}. \qquad \boxtimes$$

<u>Def. 4</u> MAXDET problem, primal standard form

$$p^* = \sup \langle (C,c), (X,s) \rangle$$
$$(X,s) \in \mathcal{D}^n$$
$$\langle (A_j, a_j), (X,s) \rangle = b_j, \quad j \in [m].$$

If $(C,c) = (0,1)$ and $a_j = 0$, then the above problem simplifies

$$p^* = \sup s \qquad\qquad\qquad p^* = \sup (\det X)^{1/n}$$
$$X \succeq 0, \ (\det X)^{1/n} \geq s \quad \Longleftrightarrow \quad X \succeq 0$$
$$\langle A_j, X \rangle = b_j, \ j \in [m] \qquad\qquad \langle A_j, X \rangle = b_j, \ j \in [m].$$

<u>Def. 5</u> MAXDET problem, dual standard form

$$d^* = \inf \sum_{j=1}^m y_j b_j$$
$$\sum_{j=1}^m y_j (A_j, a_j) - (C,c) \in (\mathcal{D}^n)^*.$$

This problem simplifies, when setting $(C, c) = (0, 1)$, $a_j = 0$, as follows:

$$d^* = \inf \sum_{j=1}^{m} y_j b_j$$
$$\sum_{j=1}^{m} y_j A_j - C \succeq 0$$
$$\det\Big(\sum_{j=1}^{m} y_j A_j - C\Big)^{1/n} \geq \frac{1}{m}.$$

Of course, the duality theory developed in Chapter III.5 also holds here.

Especially useful for MAXDET: optimality condition.

Theorem 6 Suppose $p^* = d^*$ and suppose (X, s) is feasible for the primal MAXDET problem and $(y_1, \dots, y_m)$ is feasible for the dual MAXDET problem. Then, (X, s), $(y_1, \dots, y_m)$ are both optimal if and only if the following three condition hold:

(i) $\cancel{\Big(\sum_{j=1}^{m} y_j A_j - C\Big)^{-1} = X}$ $\quad X\Big(\sum_{j=1}^{m} y_j A_j - C\Big) = \alpha I$ for some $\alpha > 0$

(ii) $s = (\det X)^{1/n}$

(iii) $-\dfrac{\sum y_j a_j - c}{n} = (\det Y)^{1/n}$.

Proof follows immediately from Theorem III.5.2 and the equality condition in Lemma 3.

$$
\begin{aligned}
0 &= \langle (X,s), (\sum_{j=1}^{m} y_j A_j - C, \sum_{j=1}^{m} y_j a_j - c) \rangle \\
&= \operatorname{Tr}\left(X\left(\sum_{j=1}^{m} y_j A_j - C\right)\right) + s\left(\sum_{j=1}^{m} y_j a_j - c\right) \\
&\geq n \left(\det X\right)^{1/n} \left(\det\left(\sum y_j A_j - C\right)\right)^{1/n} + s\left(\sum_{j=1}^{m} y_j a_j - c\right) \\
&\geq n \; s \left(-\frac{\sum y_j a_j - c}{n}\right) + s\left(\sum y_j a_j - c\right) \\
&= 0.
\end{aligned}
$$

The equality case for the first inequality yields (i).
The equality case for the second inequality yields (ii) and (iii).