§ 3 Approximation of polytopes by ellipsoids

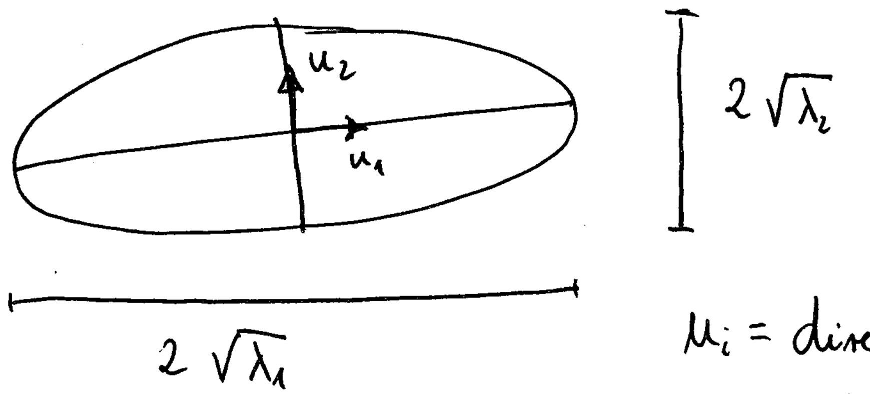
Recall:

Def. 1 Matrix $A \in S^n_{\succ 0}$, vector $x \in \mathbb{R}^n$ define the ellipsoid

$$\mathcal{E}(A, x) = \{ y \in \mathbb{R}^n : (y-x)^T A^{-1} (y-x) \leq 1 \}.$$

Ex.: $\mathcal{E}(r^2 I_n, 0) = r B_n$, where $B_n = \{ y \in \mathbb{R}^n : \|y\| \leq 1 \}$ n-dimensional unit ball.

Geometric properties can be read off from the spectral decomposition of $A = \sum_{i=1}^{n} \lambda_i u_i u_i^T$.



u_i = direction of axis.

$$\operatorname{vol} \mathcal{E}(A, x) = \sqrt{\det A} \underbrace{\operatorname{vol} B_n}_{\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}}$$

Ellipsoid is an affine image of the unit ball:

$$\mathcal{E}(A^2, x) = \{ Ay + x : y \in B_n \}$$

Theorem 2 (inner approximation)

Let $\mathcal{E}(A^2, x)$ be an ellipsoid and let

$$P = \{ y \in \mathbb{R}^n : a_1^T y \le b_1, \dots, a_m^T y \le b_m \}$$

be a polytope. We have

$$\mathcal{E}(A^2, x) \subseteq P \iff \| A a_j \| \le b_j - a_j^T x \,, \quad j \in [m]$$
$$\iff (A a_j, b_j - a_j^T x) \in \mathcal{L}^{n+1}.$$

Proof

$$\forall y \in \mathbb{R}^n : \|y\| \le 1 : \quad a_j^T (Ay + x) \le b_j \,, \quad j \in [m]$$

$$\iff \max_{\substack{y \in \mathbb{R}^n \\ \|y\| \le 1}} a_j^T A y \le b_j - a_j^T x \,, \quad j \in [m]$$

Cauchy-Schwarz gives the maximum:

$$\max_{\|y\| \le 1} (A a_j)^T y = (A a_j)^T \frac{A a_j}{\|A a_j\|} = \| A a_j \| ,$$

and the claim follows. $\boxtimes$

Goal: Find best inner approximation for P, i.e.

$\mathcal{E}(A^2, x)$ with $\mathcal{E}(A^2, x) \subseteq P$ and $\underset{\det A \, \mathrm{vol}\, B_n.}{\underset{\|}{\mathrm{vol}\, \mathcal{E}(A^2, x)}}$ maximal.

Conic programming formulation:

$$\max s$$
$$(A,s) \in \mathcal{D}^n, \quad x \in \mathbb{R}^n$$
$$(Aa_j, b_j - a_j^T x) \in \mathcal{L}^{n+1}, \quad j \in [m]$$

$$= \max s$$
$$(A,s) \in \mathcal{D}^n, \quad x \in \mathbb{R}^n, \quad (y_j, t_j) \in \mathcal{L}^{n+1}, \quad j \in [m]$$
$$y_j = Aa_j, \quad t_j = b_j - a_j^T x, \quad j \in [m].$$

Remarks: (i) We work with the cone $\mathbb{R}^n$ here which is not pointed. This is more convenient and does not cause problems. In principle one could write $x = x^+ - x^-$, $x^+ \in \mathbb{R}^n_{\geq 0}$, $x^- \in \mathbb{R}^n_{\geq 0}$.

(ii) The conic program has a unique solution because the function $F(X) = -(\det X)^{1/n}$ is strictly convex on line segments $[X,Y]$ with $X \neq \alpha Y$. (follows from the equality case of the AM-GM inequality, Lemma 1.6 (b)). The optimal ellipsoid is uniquely determined.

Def. 3 Let P be a polytope. The ellipsoid E with $E \subseteq P$ having largest volume is called Loewner-John ellipsoid $\mathcal{E}_{in}(P)$ of P.

Theorem 4 (outer approximation)

Let $\mathcal{E}(A,x)$ be an ellipsoid and let $P = \operatorname{conv}\{x_1, \dots, x_N\} \subseteq \mathbb{R}^n$ be a polytope. We have

$$P \subseteq \mathcal{E}(A,x) \iff \exists s \in \mathbb{R} : \begin{pmatrix} s & d^T \\ d & A^{-1} \end{pmatrix} \in \mathcal{S}^{n+1}_{\succeq 0}, \quad d = A^{-1}x$$

$$\text{and} \quad x_i^T A^{-1} x_i - 2x_i^T d + s \leq 1, \quad i \in [N].$$

<u>Proof</u> For verifying the condition $P \subseteq \mathcal{E}(A,x)$ it suffices to consider only the points x_i, $i \in [N]$.

$$\begin{aligned}
x_i \in \mathcal{E}(A,x) &\iff (x_i - x)^T A^{-1}(x_i - x) \leq 1 \\
&\iff x_i^T A^{-1} x_i - 2x_i^T A^{-1} x + x^T A^{-1} x \leq 1 \\
(\text{use } d = A^{-1}x) &\iff x_i^T A^{-1} x_i - 2x_i^T d + d^T A d \leq 1 \\
&\iff x_i^T A^{-1} x_i - 2x_i^T d + s \leq 1
\end{aligned}$$

with $s \geq d^T A d$.

Condition $s \geq d^T A d$ is equivalent to $\begin{bmatrix} s & d^T \\ d & A^{-1} \end{bmatrix} \succeq 0$ by the Schur complement (Prop. III.2.8) because A is positive definite. $\boxtimes$

Goal: Find best outer approximation of P, i.e. $\mathcal{E}(A,x)$ with $P \subseteq \mathcal{E}(A,x)$ and $\mathrm{vol}(\mathcal{E}(A,x)) = \sqrt{\det A}\,\mathrm{vol}\,B_n$ minimal.

Conic programming formulation:

$$\max\ (\det A^{-1})^{1/n}$$
$$\begin{pmatrix} s & d^T \\ d & A^{-1} \end{pmatrix} \in S^{n+1}_{\succeq 0}$$
$$x_i^T A^{-1} x_i - 2x_i^T d + s \leq 1, \quad i \in [N]$$

$$= \max\ t$$
$$(B,t) \in \mathcal{D}^n, \quad \begin{pmatrix} s & d^T \\ d & B \end{pmatrix} \in S^{n+1}_{\succeq 0}$$
$$s_1, \dots, s_N \geq 0$$
$$x_i^T B x_i - 2x_i^T d + s + s_i = 1, \quad i \in [N]$$

$$= \max\ t$$
$$(B,t) \in \mathcal{D}^n, \quad Y \in S^{n+1}_{\succeq 0}, \quad s \in \mathbb{R}^N$$
$$\langle E_{ij}, B \rangle + \langle -E_{i+1,j+1}, Y \rangle = 0 \qquad 1 \leq i \leq j \leq n$$
$$\left\langle \begin{bmatrix} 1 & -x_i^T \\ -x_i & x_i x_i^T \end{bmatrix}, Y \right\rangle + s_i = 1, \qquad i \in [N].$$

Def. 5 Let P be a polytope. The ellipsoid E with $P \subseteq E$ having smallest volume is called Loewner–John ellipsoid $\mathcal{E}_{out}(P)$ of P.

Consider the dual MAXDET program

$$\min \sum_{i=1}^{N} y_i$$

$$\left(\sum_{ij} z_{ij} E_{ij}, -1 \right) \in (\mathcal{D}^n)^*$$

$$- \sum_{ij} z_{ij} E_{i+1,j+1} + \sum_i y_i \begin{bmatrix} 1 & -x_i^T \\ -x_i & x_i x_i^T \end{bmatrix} \succeq 0$$

$$y_i \geq 0, \qquad i \in [N].$$

<u>Lemma 6</u> Both programs are strictly feasible, therefore strong duality holds, if $\dim P = n$.

<u>Proof</u> <u>Strict feasibility of the primal</u>: Since P is a polytope, P is bounded. So there is a ball $B(x,r)$ with center x and radius r so that $P \subseteq B(x,r)$. Make r so big that there are no vertices of P lying on the boundary of $B(x,r)$. This defines a strictly feasible solution of the primal: For $\varepsilon > 0$ sufficiently small define

$$B = \frac{1}{r^2} I_n, \quad t = \varepsilon$$

$$Y = \begin{bmatrix} s & d^T \\ d & B \end{bmatrix}, \quad d = Bx, \quad s = d^T B^{-1} d + \varepsilon$$

$$s_i = 1 - x_i^T B x_i - 2x_i^T d + s.$$