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# Convex Optimization

Winter Term 2015/16

**— Exercise Sheet 13 —**

**Exercise 13.1.** Let $G = (V, E)$ be a graph. Determine the dual of the following semidefinite program and prove that strong duality holds

$$\begin{aligned} \vartheta^+(G) = \max \quad & \langle J, X\rangle \\ & X \succeq 0 \\ & \mathrm{Tr}(X) = 1 \\ & X_{i,j} = 0 \text{ for } \{i,j\} \in E \\ & X_{i,j} \leq 0 \text{ for } \{i,j\} \notin E. \end{aligned}$$

**Exercise 13.2.** Let $G = (V, E)$ be *regular*, i.e. $e$ is an eigenvector of the adjacency matrix $A_G$ of $G$. Show:

$$\vartheta(G) \leq |V| \frac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}},$$

where $\lambda_{\min}$ is the smallest and $\lambda_{\max}$ is the largest eigenvalue of $A_G$.

**Exercise 13.3.** Let $G = (V, E)$ be a graph on $n$ vertices. A set of vectors $u_1, \ldots, u_n \in \mathbb{R}^d$ is called a $d$-dimensional vector labeling of $G$ if $\|u_i\| = 1$ for all $i \in V$ and satisfy $u_i^{\mathsf{T}} u_j = 0$ whenever $\{i,j\} \in E$. Let $\xi(G)$ be the smallest $d$ so that $G$ has a $d$-dimensional vector labeling. Show:

$$\vartheta(\overline{G}) \leq \xi(G) \leq \chi(G).$$

**Exercise 13.4.** Determine the Shannon capacity of all graphs having four vertices.

**Hand-in:** Until Tuesday, 2nd February, 2pm at the "Convex optimization" mailbox in room 3.01 (Studierendenarbeitsraum) of the Mathematical Institute. Please add your name, student number, and group number to your solution sheet.