Strict feasibility of the dual:

Since $\dim P = n$ we may assume, after performing an affine transformation that $x_1 = e_1, \dots, x_n = e_n$, $x_{n+1} = 0$.

Set z_{ij} so that $\sum_{i,j} z_{ij} E_{ij} = I_n$. Then $(I_n, -1) \in \operatorname{int} (\mathcal{D}^n)^*$.

For $\varepsilon > 0$ set $y_1 = \dots = y_n = 2$, $y_{n+1} = 2n + \varepsilon$.

$$y_{n+2} = \dots = y_N = \varepsilon.$$

Then

$$- I_n + \sum_{i=1}^{N} y_i \begin{bmatrix} 1 & -x_i \\ -x_i & x_i x_i^T \end{bmatrix}$$

$$= \underbrace{\varepsilon \sum_{i=n+2}^{N} \begin{bmatrix} 1 \\ -x_i \end{bmatrix} \begin{bmatrix} 1 \\ -x_i \end{bmatrix}^T}_{\geq 0} + \underbrace{\sum_{i=1}^{n} \left(2 \begin{bmatrix} 1 \\ -e_i \end{bmatrix} \begin{bmatrix} 1 \\ -e_i \end{bmatrix}^T - \begin{bmatrix} 0 \\ e_i \end{bmatrix} \begin{bmatrix} 0 \\ e_i \end{bmatrix}^T \right)}_{\begin{bmatrix} 2n & -2 & \dots & -2 \\ -2 & 1 & & \\ \vdots & & \ddots & \\ -2 & & & 1 \end{bmatrix}}$$

$$+ \begin{bmatrix} 2n+\varepsilon & 0 \\ 0 & 0 \end{bmatrix}$$

This matrix is positive definite: Use Schur complement:

$$(-2e)^T I_n (-2e) = 4n < 4n + \varepsilon. \qquad \boxtimes$$

Theorem 7 (John's optimality condition, 1948)

Let $P \subseteq \mathbb{R}^n$ be a polytope with $\dim P = n$. We have that the following statements are equivalent:

(i) $\mathcal{E}_{out}(P) = B_n$ (B_n, the n-dimensional unit ball, $B_n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$)

$\Longleftrightarrow$ $P \subseteq B_n$, $\exists\, \lambda_1, \dots, \lambda_M > 0$, $x_1, \dots, x_M$ vertices of P so that

(a) $\|x_i\| = 1$, $i \in [M]$

(b) $\sum_{i=1}^{M} \lambda_i x_i = 0$

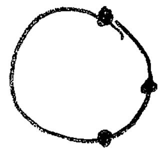
(c) $\sum_{i=1}^{M} \lambda_i x_i x_i^T = I_n$.

(ii) $\mathcal{E}_{in}(P) = B_n \Longleftrightarrow B_n \subseteq P$, $\exists\, \lambda_1, \dots, \lambda_M > 0$, $x_1, \dots, x_M \in \partial P$ so that (a) - (c) holds.

Remarks * By applying an affine transformation one can always assume that the Loewner-John ellipsoid is the unit ball.

* Historically, the John optimality conditions were among the first optimality conditions of a nonlinear program.

* Condition (b) says that not all points $x_1, \dots, x_M$ lie on one side of the unit sphere S^{n-1}, i.e.

 is not possible.

* Condition (c) says that $x_1, \dots, x_M$ behave similar to an orthonormal basis: For $x, y \in \mathbb{R}^n$

$$x^T y = x^T I_n y = \sum_{i=1}^{M} \lambda_i (x^T x_i)(x_i^T y).$$

* Condition (b) and (c) imply: $M \geq n+1$.

Proof (ii) follows from (i) by considering the polar polytope

$$P^* = \{ y \in \mathbb{R}^n : x^T y \leq 1 \text{ for all } x \in P \}.$$

$\rightarrow$ exercise.

(i) "$\Longleftarrow$": Application of weak duality:

Consider the dual program (from page 90). Define

$$(y_1, \dots, y_M, y_{M+1}, \dots, y_N) = \left(\frac{\lambda_1}{n}, \dots, \frac{\lambda_M}{n}, 0, \dots, 0 \right).$$

$$\sum z_{ij} E_{ij} = \frac{1}{n} I_n.$$

This is a feasible solution for the dual (check it!).