Consider the primal program (page 89). Define

$$(B,t) = (I_n, 1), \quad Y = \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix}.$$

This is a feasible solution for the primal because $P \subseteq B_n$. The value of the primal is equal to 1. The value of the dual is equal to $\frac{1}{n} \sum_{i=1}^{M} \lambda_i = 1$, too, because

$$\sum \lambda_i \overset{(i)}{=} \sum \lambda_i x_i^T x_i = Tr\left(\sum \lambda_i x_i x_i^T\right) \overset{(iii)}{=} Tr(I) = n.$$

"$\Longrightarrow$" Application of strong duality:

· $P \subseteq B_n$: ✓

Let $y_1^*, \ldots, y_N^*$ be an optimal solution of the dual program. After reordering we may assume

$$y_1^* > 0, \ldots, y_M^* > 0, \quad y_{M+1}^* = \ldots = y_N^* = 0.$$

By assumption, $(B,t) = (I_n, 1)$, $Y = \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix}$ is an optimal solution of the primal. By complementary slackness (Theorem III.5.2) we know $s_i = 0$, $i = 1, \ldots, M$. Hence

$$\left\langle \begin{bmatrix} 1 & -x_i^T \\ -x_i^T & x_i x_i^T \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \right\rangle = 1 \quad \text{for } i \in [M].$$

This shows $\|x_i\| = 1$, condition (a).

Let $Z^* = \sum z_{ij}^* E_{ij}$ be an optimal solution for the dual. Then (Theorem 2.6): $I_n Z^* = \alpha I_n$ for $\alpha > 0$, and $(\det Z^*)^{1/n} = \frac{1}{n}$. Hence, $Z^* = \frac{1}{n} I_n$.

Let Y^* be optimal for the primal. Then

$$Y^* \left(- \sum_{i,j} z_{ij}^* E_{i+1,j+1} + \sum_i y_i^* \begin{bmatrix} 1 & -x_i^T \\ -x_i & x_i x_i^T \end{bmatrix} \right) = 0$$

$$\Longleftrightarrow \sum_i y_i^* \begin{bmatrix} 1 & -x_i^T \\ -x_i & x_i x_i^T \end{bmatrix} = \begin{bmatrix} * & 0 \\ 0 & \frac{1}{n} I_n \end{bmatrix}.$$

(because $\begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$)

Define $\lambda_i = n y_i^*$, then (b), (c) follow from the equality above. $\boxtimes$

<u>Corollary 8</u> Let $P \subseteq \mathbb{R}^n$ be a polytope with $\dim P = n$.

(i) If $\mathcal{E}_{out}(P) = B_n$, then $\frac{1}{n} B_n \subseteq P \subseteq B_n$.

(ii) If $\mathcal{E}_{in}(P) = B_n$, then $B_n \subseteq P \subseteq n B_n$.

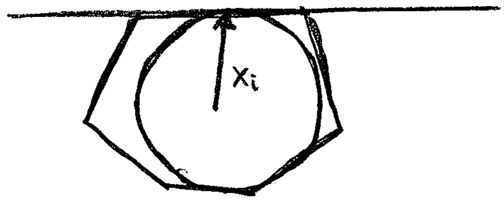
Both inclusions are best possible $\to$ Exercises.

<u>Proof</u> (i) follows from (ii) by polarity.

(ii) Apply Theorem 7: We have

$$\sum_{i=1}^{M} \lambda_i x_i = 0, \quad \sum_{i=1}^{M} \lambda_i x_i x_i^T = I_n, \quad \lambda_i > 0.$$

Consider the supporting hyperplane through $x_i \in B_n \cap \partial P$.



It is orthogonal to x_i.

Hence,

$$B_n \subseteq P \subseteq Q = \{x \in \mathbb{R}^n : x_i^T x \le 1, \, i \in [M]\}.$$

If $x \in Q$, then $x^T x_i \in [-\|x\|, 1]$. Hence,

$$0 \le \sum_{i=1}^{M} \lambda_i \, (1 - x^T x_i)(\|x\| + x^T x_i)$$

$$= \|x\| \sum_{i=1}^{M} \lambda_i + (1 - \|x\|) \sum_{i=1}^{M} \lambda_i x^T x_i - \sum_{i=1}^{M} \lambda_i (x^T x_i)^2$$

$$= \|x\| \, n \quad + \quad 0 \quad - \quad \|x\|^2$$

$\Longrightarrow \|x\| \le n.$

because $\sum \lambda_i = \sum \lambda_i x_i^T x_i$

$= \mathrm{Tr}\left(\sum \lambda_i x_i x_i^T\right)$

$= \mathrm{Tr}(I_n) = n.$

$\boxtimes$