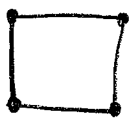
Chapter VII Packings and colorings in graphs

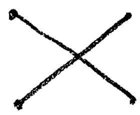
§ 1 Basic definition

Let $G = (V, E)$ be an undirected graph. Its complementary graph $\overline{G} = (V, \overline{E})$ is defined by

$$\overline{E} = \{ \{i,j\} : i,j \in V,\ i \neq j,\ \{i,j\} \notin E \}.$$



G



$\overline{G}$

A set $I \subseteq V$ is called independent (stable) if for all $i,j \in I$: $\{i,j\} \notin E$. The independence (stability) number of G is

$$\alpha(G) = \max \{ |I| : I \subseteq V \text{ is independent} \}.$$

A set $C \subseteq V$ is called a clique if for all $i,j \in C$, $i \neq j$: $\{i,j\} \in E$.

The clique number of G is

$$\omega(G) = \max \{ |C| : C \subseteq V \text{ is a clique} \} = \alpha(\overline{G}).$$

A *k-coloring* of G, $k \in \mathbb{N}$, is a map $\varphi : \{1, \dots, k\} \to V$ with $\varphi(i) \neq \varphi(j)$ for all $\{i,j\} \in E$.

A k-coloring is a partition $V = I_1 \dot\cup \dots \dot\cup I_k$ into k independent sets.

The *chromatic number* of G is

$$\chi(G) = \min \{ k : \exists\ k\text{-coloring } \varphi \text{ of } G \}.$$

The decision problems

- $\alpha(G) \geq k$?
- $\omega(G) \geq k$?
- $\chi(G) \leq k$?

are all NP-complete.

Even the restricted problem: Is "$\chi(G) \leq 3$?" for planar graphs G is NP-complete.

[On the other hand: $\chi(G) \leq 4$ for every planar graph G. (4-color theorem of Appel and Haken (1976))].

Easy: "$\chi(G) \leq 2$?" In other words: Is G bipartite?

§2 Semidefinite relaxation for α and χ

Def. 1 Let $G = (V, E)$. The optimal solution of the following semidefinite program is called the Lovász theta number of G.

$$\begin{aligned} \vartheta(G) = \max\ & \langle J, X \rangle \\ & X \succeq 0 \quad (X \in S^V_{\succeq 0}) \\ & \operatorname{Tr}(X) = 1 \\ & X_{ij} = 0 \quad \text{for } \{i,j\} \in E. \end{aligned}$$

Remark $\vartheta(G)$ = semidefinite relaxation of a quadratic program with 0/1-constraints which computes $\alpha(G)$ ($\rightarrow$ see Chapter IV.3).

Theorem 2 (Lovász' Sandwich Theorem)

$$\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G}).$$

Proof $\alpha(G) \leq \vartheta(G)$: Let $I \subseteq V$ be an independent set with $|I| = \alpha(G)$. Define its characteristic vector $\chi^I \in \mathbb{R}^V$ by

$$(\chi^I)_i = \begin{cases} 1 & \text{if } i \in I, \\ 0 & \text{otherwise,} \end{cases}$$

and the matrix

$$X = \frac{1}{|I|} \chi^I (\chi^I)^T \in S^V.$$

Then (check!) X is a feasible solution for the semidefinite program with $\langle J, X \rangle = |I|$. Hence, $\alpha(G) \leq \vartheta(G)$.

<u>$\vartheta(G) \leq \chi(\overline{G})$:</u> Let X be a feasible solution and let $V = C_1 \dot\cup \ldots \dot\cup C_k$ be a partition into k cliques.

<u>Claim</u>: $\langle J, X \rangle \leq k \quad (\Rightarrow \vartheta(G) \leq \chi(\overline{G}))$.

<u>Proof</u>: We have $\sum_{i=1}^{k} \chi^{C_i} = e = (1, \ldots, 1)^T$ since we have a partition.

Furthermore,

$$
\begin{aligned}
0 \leq Y &= \sum_{i=1}^{k} (k\chi^{C_i} - e)(k\chi^{C_i} - e)^T \\
&= k^2 \sum_{i=1}^{k} \chi^{C_i}(\chi^{C_i})^T - e \underbrace{\left(\sum_{i=1}^{k} k\chi^{C_i}\right)^T}_{ke^T} - \underbrace{\left(\sum_{i=1}^{k} k\chi^{C_i}\right)}_{ke} e^T \\
&\quad + \sum_{i=1}^{k} ee^T \\
&= k^2 \sum_{i=1}^{k} \chi^{C_i}(\chi^{C_i})^T - kJ, \qquad J = ee^T,
\end{aligned}
$$

and

$$0 \le \langle X, Y \rangle = k^2 \langle X, \sum_{i=1}^{k} \chi^{C_i} (\chi^{C_i})^T \rangle - k \langle J, X \rangle$$

$$= k^2 \operatorname{Tr}(X) - k \langle J, X \rangle$$

C_i's are cliques, $X_{vw} = 0$ if $\{v, w\} \in E$

$$= k^2 - k \langle J, X \rangle$$

$$\Longrightarrow \langle J, X \rangle \le k. \qquad \boxtimes$$