§3 Perfect Graphs

Def. 1 Let $G=(V,E)$ be a graph and $S \subseteq V$.
$G[S] = (S, \{\{i,j\} \in E : i \in S, j \in S\})$
is the <u>subgraph of G induced by S.</u>

A graph H is called induced subgraph of G, if there is a set $S \subseteq V$ for which $H = G[S]$ holds. Notation: $H \subseteq G$.

Def. 2 A graph is called <u>perfect</u>, if $\omega(H) = \chi(H)$ holds for all induced subgraphs $H \subseteq G$.

Ex. 3

a) Bipartite graphs are perfect ($\omega(H) = \chi(H)$ clear $(=2)$)

b) C_{2n+1}, $\overline{C_{2n+1}}$ are not perfect

c) Line graphs of bipartite graphs G, denoted by $L(G)$ are perfect, where $L(G) = (E, \{\{e_1, e_2\} : e_1 \cap e_2 \neq \emptyset\})$.

⌜Prove by using: $\chi(\overline{G}) \hat{=}$ min # cliques covering $V(G)$⌟

"Strong perfect graph theorem" (Chudnovsky, Robertson, Seymour, Thomas 2006)

A graph is perfect if and only if neither C_{2n+1} nor $\overline{C_{2n+1}}$ are induced subgraphs.

Proof: approx. 180 pages long.

Theorem 4: ("perfect graph theorem", Lovász 1972)

G is perfect $\Leftrightarrow$ $\bar{G}$ is perfect

Proof: Show: G is perfect $\Leftrightarrow$ For each induced subgraph $H \subseteq G$ the inequality $|V(H)| \leq \omega(H) \cdot \alpha(H)$ holds.

[Because then $\omega(H) = \alpha(\bar{H})$ and $\alpha(H) = \omega(\bar{H})$ imply Theorem 4]

"$\Rightarrow$": Let G be perfect and $H \subseteq G$. Then $\chi(H) = \omega(H)$, i.e. $V(H)$ can be covered by $\omega(H)$ independent sets.

Thus $|V(H)| \leq \omega(H)\,\alpha(H)$.

"<=" Assume G satisfies the inequalities

$|V(H)| \leq \omega(H)\alpha(H)$, but G is not perfect.

Choose G with $|V(G)|$ minimal, i.e.

$\omega(G) < \chi(G)$ and $\omega(H) = \chi(H)$ holds for each induced subgraph $H \subseteq G$ with $H \neq G$.

1. Claim: $\exists\ \alpha(G)\omega(G) + 1$ independent sets $S_0, \dots, S_{\alpha(G)\omega(G)}$, such that each node in G is covered by exactly $\alpha(G)$ of those sets.

Proof: Let S_0 be an independent set with $|S| = \alpha(G)$.

$\forall v_0 \in S_0$: $G[V \setminus \{v_0\}]$ is perfect (G was chosen to be minimal).

Then $\chi(G[V\setminus\{v_0\}]) = \omega(G[V\setminus\{v_0\}]) \leq \omega(G)$.

Thus it is possible to partition $V\setminus\{v_0\}$ into $\omega(G)$ independent sets. If we continue to do this for each of the $\alpha(G)$ nodes in S_0 we get $\alpha(G)\omega(G)$ independent sets $S_1, \dots, S_{\alpha(G)\omega(G)}$, which by additionally considering S_0 satisfy the claim.

2. Claim: $\forall i = 0, \ldots, \alpha(G)\,\omega(G)$ $\exists K_i$ clique, $|K_i| = \omega(G)$ with $K_i \cap S_i = \emptyset$ and $K_i \cap S_j \neq \emptyset$ for $j \neq i$.

Proof: Since $G[V \setminus S_i]$ is perfect,
$\chi(G[V\setminus S_i]) = \omega(G[V\setminus S_i]) \leq \omega(G)$ holds.

Thus $\chi(G[V\setminus S_i]) = \omega(G)$ must be true, because if $\chi(G[V\setminus S_i]) \leq \omega(G) - 1$, we could colour G with $\omega(G)$ colours.

Thus we have $\omega(G[V\setminus S_i]) = \omega(G)$ and there is a clique K_i with $K_i \cap S_i = \emptyset$ and $|K_i| = \omega(G)$.
Furthermore $K_i \cap S_j \neq \emptyset$ for $j \neq i$, because each of the $\omega(G)$ elements of K_i belongs to $\alpha(G)$ independent sets S_j (Claim 1) and $|K_i \cap S_j| \leq 1$, thus $|K_i \cap S_j| = 1$.
Define matrices $M, N \in \mathbb{R}^{|V| \times (\alpha(G)\,\omega(G)+1)}$:

$$M = [\chi^{S_0} \ldots \chi^{S_{\alpha(G)\omega(G)+1}}]$$

$$N = [\chi^{K_0} \ldots \chi^{K_{\alpha(G)\omega(G)+1}}]$$

Then $M^T N = J - I$ holds due to Claim 2.

Recall that $J-I$ is a regular matrix, i.e.

$$\operatorname{rank}(M^T N) = \operatorname{rank}(J-I) = \alpha(G)\,\omega(G) + 1.$$

But on the other hand

$$\operatorname{rank}(M^T N) \leq \operatorname{rank}(N) \leq V(G).$$

This implies $\alpha(G)\,\omega(G) < V(G)$. $\lightning$

<u>Corollary 5</u> Let G be perfect. Then

$$\alpha(G) = \vartheta(G) = \chi(\bar{G}).$$

<u>Proof</u> By using $\overline{\text{VII}}$ §2 Thm. 2 and Theorem 4 we obtain

$$\alpha(G) \leq \vartheta(G) \leq \chi(\bar{G}) = \omega(\bar{G}) = \alpha(G),$$

which means equality in each case.

This means one can determine α, ω, χ in perfect graphs efficiently.

Now: Find $S \subseteq V$ with $|S| = \alpha(G)$ respectively find partition $V = S_1 \cup \ldots \cup S_k$ with $k = \chi(G)$.