Thm 6: Let $G=(V,E)$ be perfect. We can efficiently find an independent set $S \subseteq V$ with $|S| = \alpha(G)$.

Proof: Given: $G=(V,E)$ with $V = \{i_1, \dots, i_n\}$
Want: $S \subseteq V$ indep. with $|S| = \alpha(G)$

$G_0 = G$
for $j = 1$ to n
  compute $\alpha(G_{j-1} \setminus i_j)$
  if $\alpha(G_{j-1} \setminus i_j) = \alpha(G)$
    $G_j = G_{j-1} \setminus i_j$
  else
    $G_j = G_{j-1}$

$S = V(G_n)$

This is efficient for perfect G, since we can compute $\alpha(G_{j-1} \setminus i_j)$ by computing $\vartheta(G_{j-1} \setminus i_j)$ in this case. In total, we compute ϑ $(n+1)$-times.

Claim: $|S| = \alpha(G)$ and S is independent.

Proof: Clearly $|S| \geq \alpha(G)$, since $\alpha(G_j) = \alpha(G)$ for all $j \in \{1, \dots, n\}$.

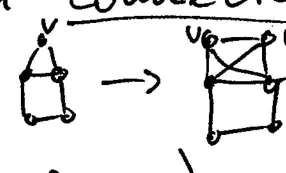
Assume S is not independent, then there is some $i_j \in S$ with $\alpha(G_n \setminus i_j) = \alpha(G)$.
But G_n is an induced subgraph of G_{j-1}, thus $\alpha(G) = \alpha(G_n \setminus i_j) \leq \alpha(G_{j-1} \setminus i_j) \leq \alpha(G)$, and hence $\alpha(G_{j-1} \setminus i_j) = \alpha(G)$.
But this means i_j would have been removed in iteration j. ↯

□

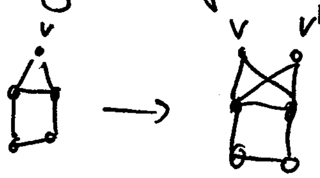
Efficient calculation of a $\chi(G)$-coloring:

Def 7: Let $G = (V, E)$ be a graph and let $v \in V$.

Then
$$H = (V \cup \{v'\},\ E \cup \{\{v, v'\}\} \cup \{\{v', w\} : \{v, w\} \in E\})$$
is the graph we get from connected duplication of v;



$$H' = (V \cup \{v'\},\ E \cup \{\{v', w\} : \{v, w\} \in E\})$$
is the graph we get from duplication of v.



Thm 8: Let G be perfect and H given by connected duplication of a vertex of G. Then H is perfect.

Proof: We show that $\overline{H}$ is perfect, from which the claim follows by the perfect graph theorem (Theorem 4).

1) $\alpha(H) = \chi(\overline{H})$:

By construction, $\alpha(H) = \alpha(G) = \chi(\overline{G})$.

Let $V = C_1 \dot\cup \ldots \dot\cup C_t$ be a partition of V into cliques.

Suppose $v \in C_1$. Then $V \cup \{v'\} = (C_1 \cup \{v'\}) \dot\cup \ldots \dot\cup C_t$ is a partition of $V \cup \{v'\}$ into cliques.

Thus, $\chi(\overline{H}) \leq \chi(\overline{G}) \underset{\overline{G} \subseteq \overline{H}}{\leq} \chi(\overline{H})$

and therefore $\alpha(H) = \chi(\overline{H})$.

2) $\alpha(H') = \chi(\overline{H'})$ for all $H' \subseteq H$.

Identical argument as above when $v' \in V(H')$.

If $v' \notin V(H')$, then $H' \subseteq G$ and $\alpha(H') = \chi(\overline{H'})$ since G is perfect.

$\square$

Def 9: Let $G=(V,E)$ be a graph and $w: V \to \mathbb{Z}_{\geq 0}$ a weight-function.
The weighted independence number is

$$\alpha_w(G) = \max\left\{ \sum_{v \in S} w(v) \;:\; S \subseteq V \text{ indep. in } G \right\}.$$

Note that $\alpha_w(G) = \alpha(G_w)$, where G_w is the graph we get from duplicating every $v \in V$ $w(v)$-times.
Also note that if H is the graph we get from connected duplication of v in $\overline{G}$, and H' is the graph we get from duplication in G, then $\overline{H} = H'$.
Thus, if G is perfect, then H' is also perfect.

Lemma 10: In a perfect graph G there is an independent set S that has nonempty intersection with all cliques C with $|C| = \omega(G)$

Proof: Let $V = S_1 \dot\cup \ldots \dot\cup S_{\omega(G)}$ be an optimal coloring of G. Then

$$\omega(G[V \setminus S_i]) = \chi(G[V \setminus S_i]) = \omega(G) - 1$$

for every S_i $\square$

Idea for an algorithm: Find such an independent set S and color $G[V\setminus S]$ recursively with $\omega(G[V\setminus S]) = \omega(G)-1$ colors.

Thm 11: Let $G=(V,E)$ be perfect. We can efficiently compute a partition of $V = S_1 \dot\cup \ldots \dot\cup S_{\chi(G)}$ into independent sets S_i.

Proof: Algorithm 1: Input: $G=(V,E)$ perfect graph.
Output: $\chi(G)$-coloring $V = S_1 \dot\cup \ldots \dot\cup S_{\chi(G)}$.

$i = 1$
while $V(G) \neq \emptyset$ do
  S_i = Result of Algorithm 2
  $G = G[V(G)\setminus S_i]$
  $i = i+1$

Algorithm 2: Input: $G=(V,E)$ perfect graph
Output: Indep. set S with: $\forall C \subseteq V$ clique in G, $|C| = \omega(G)$: $|S \cap C| = 1$.

$t = 0$
repeat
  $w = \sum_{i=1}^{t} \chi^{c_i}$

Find indep. set $S \subseteq V$ with $\sum_{v \in S} w(v) = \alpha_w(G)$ (*)
[apply Thm 6 to G_w]

If $\omega(G[V \setminus S]) = \omega(G)$

$t = t+1$

Compute clique C_t in $G[V \setminus S]$ with $|C_t| = \omega(G)$

[apply Thm 6 to $\overline{G[V \setminus S]}$]

until $\omega(G[V \setminus S]) < \omega(G)$

output S.

The Step (*) insures that the set S in Algorithm 2 intersects all cliques $C_1, \dots, C_t$. When we find S with $\omega(G[V \setminus S]) < \omega(G)$, then S intersects all cliques C in G with $|C| = \omega(G)$.

<u>Claim:</u> The number of iterations in Algorithm 2 is at most $|V|$.

<u>Proof</u>: Consider the affine subspace

$$L_t = \{ x \in \mathbb{R}^V : \sum_{i \in C_j} x_i = 1, \ j = 1, \dots, t \}.$$

Then $L_t \supsetneq L_{t+1}$, since $\chi^S \in L_t \setminus L_{t+1}$ for the set S constructed in iteration t.

Therefore, $\dim L_{t+1} < \dim L_t$, and thus $t \leq \dim \mathbb{R}^V = |V|$.

□