Def. 8 Let $\sigma : V \to V$ be a permutation. Define the permutation matrix $P^\sigma \in \mathbb{R}^{V\times V}$ by

$$P^\sigma_{i,j} = \begin{cases} 1, & \text{if } i = \sigma(j), \\ 0, & \text{otherwise}. \end{cases}$$

Let $X \in S^V$ be a symmetric matrix. Define the left action of σ on X by

$$\sigma(X) = P^\sigma X (P^\sigma)^T = \left(X_{\sigma^{-1}(i), \sigma^{-1}(j)} \right)_{i,j\in V}.$$

Lemma 9 Let $G = (V, E)$ be a graph. Consider

$$\vartheta(G) = \max \langle J, X\rangle \quad X \in S^V_{\succeq 0},\ \mathrm{Tr}(X) = 1, \quad X_{ij} = 0 \text{ if } \{i,j\} \in E.$$

This optimisation problem has an optimal solution X^* which is invariant under $\mathrm{Aut}(G)$, i.e. $\sigma(X^*) = X^*$ for all $\sigma \in \mathrm{Aut}(G)$.

Proof Let X be an optimal solution of $\vartheta(G)$; which exists by strong duality. Then $\sigma(X)$ is also an optimal solution of $\vartheta(G)$. (check it!).

Define $X^* = \frac{1}{|\mathrm{Aut}(G)|} \sum_{\sigma \in \mathrm{Aut}(G)} \sigma(X)$, which again is an optimal solution of $\vartheta(G)$. By construction, we have $\sigma(X^*) = X^*$ for all $\sigma \in \mathrm{Aut}(G)$ because

$$\sigma(X^*) = \frac{1}{|\mathrm{Aut}(G)|} \sum_{\tau \in \mathrm{Aut}(G)} \sigma\tau(X)$$

$$= \frac{1}{|\mathrm{Aut}(G)|} \sum_{\tau \in \mathrm{Aut}(G)} \tau(X) = X^*.$$

$\boxtimes$

<u>Proposition 10</u> Let $G = (V, E)$ be a vertex transitive graph. Then there is an optimal solution X^* of $\vartheta(G)$ with

$X^*_{ii} = \frac{1}{|V|}$ for all $i \in V$, and

$X^* e = \frac{\vartheta(G)}{|V|} e$.

<u>Proof</u> Let X^* be an optimal solution of $\vartheta(G)$ which is invariant under $\mathrm{Aut}(G)$ (see ~~top~~ Lemma 9). By Ex. 12.2

$X^* e = \vartheta(G) \operatorname{diag} X^*$.

So we have to show that $\operatorname{diag} X^* = \left(\frac{1}{|V|}, \ldots, \frac{1}{|V|}\right)$.

Since G is vertex transitive we have for $i, j \in V$

$$X^*_{jj} = X^*_{\sigma^{-1}(i), \sigma^{-1}(i)} = \sigma(X^*)_{ii} = X^*_{ii}$$

for some $\sigma \in \mathrm{Aut}(G)$ with $\sigma(j) = i$. Hence all diagonal coefficients of X^* coincide. Since $\mathrm{Tr}(X^*) = 1$, we have $X^*_{ii} = \frac{1}{|V|}$. $\boxtimes$

<u>Theorem 11</u> If $G = (V, E)$ is vertex transitive, then

$$\vartheta(G)\,\vartheta(\overline{G}) = |V|.$$

<u>Proof</u> <u>$\vartheta(G)\,\vartheta(\overline{G}) \geq |V|$</u>:

Consider dual SDP

$$\vartheta(G) = \min t$$
$$Z - J \succeq 0, \quad Z_{ii} = t, \quad i \in V$$
$$Z_{ij} = 0 \quad \text{if } \{i,j\} \notin E(G)$$

Let X be optimal for $\vartheta(G)$, $\overline{X}$ be optimal for $\vartheta(\overline{G})$.

We have

$$0 \leq \langle X - J, J \rangle = \langle X, J \rangle - \underbrace{\langle J, J \rangle}_{|V|^2} \Rightarrow \langle X, J \rangle \geq |V|^2.$$

similarly, $\langle \overline{X}, J \rangle \geq |V|^2$.

Furthermore,

$$\langle X, \bar{X} \rangle = \sum_{i,j \in V} X_{ij} \bar{X}_{ij} = \sum_{i \in V} X_{ii} \bar{X}_{ii}$$

$$= |V| \vartheta(G) \vartheta(\bar{G}).$$

We also have

$$0 \leq \langle X - J, \bar{X} \rangle = \langle X, \bar{X} \rangle - \langle X, J \rangle \leq \langle X, \bar{X} \rangle - |V|^2.$$

So

$$|V|^2 \leq \langle X, \bar{X} \rangle = |V| \vartheta(G) \vartheta(\bar{G}) \Rightarrow |V| \leq \vartheta(G) \vartheta(\bar{G}).$$

[This inequality is valid for arbitrary graphs G].

<u>$\vartheta(G)\vartheta(\bar{G}) \leq |V|$:</u>

Let X^* be an optimal solution of the primal SDP $\vartheta(G)$ as in Proposition 10. Define a feasible solution Y of the dual SDP $\vartheta(\bar{G})$ as follows: $Y = \frac{|V|^2}{\vartheta(G)} X^*$, then $t = \frac{|V|}{\vartheta(G)}$. $\left(\Rightarrow \vartheta(G)\vartheta(\bar{G}) \leq \vartheta(G) \frac{|V|}{\vartheta(G)} = |V| \right)$.

Y is indeed feasible because:

$Y_{ij} = 0$ for $\{i,j\} \in E(G)$: ✓

$Y - J \succeq 0$:

$\frac{|V|^2}{\vartheta(G)} X^* - J \succeq 0$ because

$$\frac{|V|^2}{\vartheta(G)} X^* e - Je = \frac{|V|^2}{\vartheta(G)} \frac{\vartheta(G)}{|V|} e - |V| e = 0.$$

(eigenvectors of J are $\mathbb{R}e \setminus \{0\}$ and $(\mathbb{R}e)^\perp \setminus \{0\}$). ⊠

<u>Corollary 12</u> $\vartheta(C_n)\,\vartheta(\overline{C_n}) = n$.

$\Rightarrow \vartheta(C_5) = \sqrt{5}$.