Chapter VIII Copositive Programming

Until now we made the impression that convex optimization is easy ($\hat{=}$ polynomial time solvable). This is (basically) true if it comes to optimizing over the cones $\mathbb{R}^n_{\geq 0}$, $\mathcal{L}^{n+1}$, $\mathcal{S}^n_{\geq 0}$, $\mathcal{D}^n$.

But: Conic programming can be NP-hard.

§1 The completely positive and the copositive cone

Def. 1 $CP_n = \text{cone}\{xx^T : x \in \mathbb{R}^n_{\geq 0}\}$ is called the cone of completely positive matrices

Prop. 2 CP_n is a proper convex cone.

Proof:
- CP_n is convex cone: ✓
- CP_n is pointed: clear because $CP_n \subseteq \mathcal{S}^n_{\geq 0} \cap \mathbb{R}^{n(n+1)/2}_{\geq 0}$.
- $\text{int}\, CP_n \neq \emptyset$: clear because $\frac{1}{2}(e_i + e_j)(e_i + e_j)^T$, $1 \leq i \leq j \leq n$, are linearly independent and lie in CP_n.

- CP_n is closed: Let $(A_i)_{i\in\mathbb{N}}$ be a sequence with $A_i \in CP_n$ and $\lim_{i\to\infty} A_i = A$.

  By the theorem of Caratheodory there are $N = \frac{n(n+1)}{2}$ vectors $x_j^i \in \mathbb{R}^n_{\geq 0}$, $j = 1, \ldots, N$, so that $A_i = \sum_{j=1}^{N} x_j^i (x_j^i)^T$.

  <u>Claim</u> $\{x_j^i : i \in \mathbb{N}, j = 1, \ldots, N\}$ is bounded.

  <u>Proof</u> For suppose not. Then there is a $j \in \{1, \ldots, N\}$ so that $\{x_j^i : i \in \mathbb{N}\}$ is not bounded. Hence there is a coordinate $k \in \{1, \ldots, n\}$ so that $\{(x_j^i)_k : i \in \mathbb{N}\}$ is not bounded. Since $x_j^i \in \mathbb{R}^n_{\geq 0}$: $(A_i)_{kk} \geq (x_j^i)_k^2 \to \infty$ for $i \to \infty$. So the limit A cannot exist. Contradiction.

Because of the claim the sets $\{x_j^i : i \in \mathbb{N}\}$, for each j, are contained in a compact subset of $\mathbb{R}^n_{\geq 0}$. Hence there exists a converging subsequence of $(x_j^i)_{i\in\mathbb{N}}$ converging to $x_j \in \mathbb{R}^n_{\geq 0}$. Now we find a common converging subsequence for all j and get $A = \sum_{j=1}^{N} x_j x_j^T$. $\boxtimes$

Def. 3 $COP_n = \{ A \in S^n_{\geq 0} : x^T A x \geq 0 \text{ for all } x \in \mathbb{R}^n_{\geq 0} \}$.

is called the cone of copositive matrices.

Clear: $S^n_{\geq 0} + \mathbb{R}^{n(n+1)/2}_{\geq 0} \subseteq COP_n$.

(but in general $\subsetneq$).

Proposition 4 $(CP_n)^* = COP_n$.

Proof "$\subseteq$": Let $A \in (CP_n)^*$. Then for all $x \in \mathbb{R}^n_{\geq 0}$

$$0 \leq \langle A, xx^T \rangle = x^T A x \implies A \in COP_n.$$

"$\supseteq$" Let $A \in COP_n$ and let $B \in CP_n$. Then

$$B = \sum_{i=1}^{k} \lambda_i x_i x_i^T, \quad \lambda_i \geq 0, \ x_i \in \mathbb{R}^n_{\geq 0} \text{ and}$$

$$\langle A, B \rangle = \sum_{i=1}^{k} \lambda_i \langle A, x_i x_i^T \rangle = \sum_{i=1}^{k} \underbrace{\lambda_i}_{\geq 0} \underbrace{x_i^T A x_i}_{\geq 0} \geq 0.$$

$\boxtimes$

Def. 5 Given $C, A_1, \ldots, A_m \in S^n$, $b_1, \ldots, b_m \in \mathbb{R}$. This defines a copositive program

$$(P) \quad p^* = \sup \langle C, X \rangle \quad X \in CP_n, \quad \langle A_j, X \rangle = b_j, \ j \in [m]$$

$$(D) \quad d^* = \inf b^T y \quad y \in \mathbb{R}^m, \quad \sum_{j=1}^m y_j A_j - C \in COP_n$$